$$S = \{\boldsymbol{v_1},\ \ldots\ \boldsymbol{v_i},\ \ldots\ \boldsymbol{v_p}\}$$

$$x_1\boldsymbol{v_1} + \ldots + x_i\boldsymbol{v_i} + \ldots + x_p\boldsymbol{v_p} = \mathbf{0}$$

equivalent to

$$\begin{bmatrix} \boldsymbol{v_1}, \ \ldots\ \boldsymbol{v_i},\ \ldots\ \boldsymbol{v_p} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_p \end{bmatrix} = \mathbf{0}$$

$$\underset{A}{\uparrow} \qquad\qquad \underset{\boldsymbol{x}}{} \quad \underset{=\ \mathbf{0}}{\uparrow}$$

$S = \{\boldsymbol{v_1},\ \ldots\ \boldsymbol{v_i},\ \ldots\ \boldsymbol{v_p}\}$ lin. **independent**
if there is only the trivial solution

$$0\boldsymbol{v_1} + \ldots + 0\boldsymbol{v_i} + \ldots + 0\boldsymbol{v_p} = \mathbf{0}$$

$S = \{\boldsymbol{v_1},\ \ldots\ \boldsymbol{v_i},\ \ldots\ \boldsymbol{v_p}\}$ lin. **dependent**
if there there are nontrivial solutions
(that is, where not all weights c_i's are 0)

$$c_1\boldsymbol{v_1} + \ldots + c_i\boldsymbol{v_i} + \ldots + c_p\boldsymbol{v_p} = \mathbf{0}$$

(*nontrivial linear dependence relation*)

<u>Linearly Dependent or Independent Sets?</u>

$$S = \left\{ \begin{bmatrix} 1 \\ -3 \\ 5 \\ 4 \end{bmatrix}, \begin{bmatrix} -4 \\ 12 \\ -20 \\ 16 \end{bmatrix} \right\}$$

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \right\}$$

$$S = \left\{ \begin{bmatrix} \pi \\ 0 \\ \sqrt{2} \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\pi \\ 0 \\ -\sqrt{2} \\ -3 \\ 1 \end{bmatrix} \right\}$$

$$S = \left\{ \begin{bmatrix} 3 \\ 319 \\ -7 \\ 12 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 6 \\ -9 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} \sqrt{2} \\ 2 \\ 16 \\ -7 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 45 \\ 3 \end{bmatrix} \right\}$$

$$S = \left\{ \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -5 \end{bmatrix} \right\}$$

Theorem $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_p}\}$ with 2 or more vectors.

• if at least one vector $\boldsymbol{v_i}$ in S is a linear combination of the others, then S is linearly dependent

• • if S is linearly dependent, then at least one vector $\boldsymbol{v_i}$ in S is a linear combination of the others.

Why is • true ?

Suppose $\boldsymbol{v_i}$ is a linear combination of the others:

Then

$$\boldsymbol{v_i} = c_1\boldsymbol{v_1} + \ldots. + (\textit{no } \boldsymbol{v_i} \textit{ term}) + \ldots + c_p\boldsymbol{v_p}$$

so: $\mathbf{0} = c_1\boldsymbol{v_1} + \ldots + (-1)\boldsymbol{v_i} + \ldots + c_p\boldsymbol{v_p}$

therefore: S is linearly dependent.

Why is ●● true ?

Assume S linearly dependent.
Then $\exists\, c_1, \ldots, c_p$, not all 0, for which

$$c_1\boldsymbol{v_1} + \ldots + c_i\boldsymbol{v_i} + \ldots + c_p\boldsymbol{v_p} = \mathbf{0} \quad (*)$$

i) if $\boldsymbol{v_1} = \mathbf{0}$, then $\boldsymbol{v_1}$ = trivial linear combination of the others(use all 0 weights)

ii) if $\boldsymbol{v_1} \neq \mathbf{0}$, then pick the <u>rightmost nonzero</u> $\boldsymbol{c_i}$ in equation $(*)$. Then

drop out any 0 terms

$$c_1\boldsymbol{v_1} + \ldots + \boldsymbol{c_i}\boldsymbol{v_i} \;\; \left| + \ldots + \boldsymbol{c_p}\boldsymbol{v_p} \right| = \mathbf{0}$$

Note: $\boldsymbol{c_i} \neq c_1$, *or else* $c_1\boldsymbol{v_1} = \mathbf{0}$, *which is impossible since* $\boldsymbol{v_1} \neq \mathbf{0}$; *but perhaps* i *is as big as* $i = p$ (***then, no blue terms dropped***)

So $c_1\boldsymbol{v_1} + \ldots + \boldsymbol{c_i}\boldsymbol{v_i} = 0$, and

$$\boldsymbol{v_i} = -\frac{c_1}{c_i}\boldsymbol{v_1} - \ldots - \frac{c_{i-1}}{c_i}\boldsymbol{v_{i-1}}$$

So we actually showed that :

if S is linearly dependent and $v_1 \neq 0,$ then some vector v_i in S must be a linear combination of the preceding vectors in the set.