Speedy One-Day Car Rental
Example

Everything that was on the pdf files is in the handout distributed (and linked in the syllabus)

Matrix Terminology & Operations

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \ldots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \ldots & a_{2n} \\ & & \vdots & & & \\ a_{i1} & a_{i2} & \cdots & \boldsymbol{a_{ij}} & \ldots & a_{in} \\ & & \vdots & & & \\ a_{m1} & a_{n2} & \ldots & a_{1j} & \ldots & a_{mn} \end{bmatrix}$$

$$\begin{matrix} \uparrow & \uparrow & & \uparrow & & \uparrow \\ [\ \boldsymbol{a_1} & \boldsymbol{a_2} & \ldots & \boldsymbol{a_j} & \ldots & \boldsymbol{a_n}\ ] \end{matrix}$$

The entries a_{ii} form the main diagonal of A :

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \\ a_{51} & a_{52} & a_{53} \end{bmatrix}$$

Diagonal matrix : square and $a_{ij} = 0$ when $i \neq j$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & -\pi & 0 \\ 0 & 0 & \frac{13}{7} \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Zero matrix : all $a_{ij} = 0$ $\quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Scalar times a matrix : $cA = c\,[\boldsymbol{a_1}\ \boldsymbol{a_2} \ldots \boldsymbol{a_n}]$
$= [c\boldsymbol{a_1}\ c\boldsymbol{a_2} \ldots c\boldsymbol{a_n}]$

$$3\begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & 6 \\ 2 & 1 & 7 \end{bmatrix} = \begin{bmatrix} 3 & 6 & -1 \\ 0 & 15 & 18 \\ 6 & 3 & 21 \end{bmatrix}$$

Sum of two matrices (must be same size $m \times n$)

$A = [\boldsymbol{a_1}\ \boldsymbol{a_2} \dots \boldsymbol{a_n}]$, $\quad B = [\boldsymbol{b_1}\ \boldsymbol{b_2} \dots \boldsymbol{b_n}]$ where $\boldsymbol{a_i}$'s, $\boldsymbol{b_i}$'s in $\mathbb{R}^m$

$A + B = [\boldsymbol{a_1} + \boldsymbol{b_1}\ \ \boldsymbol{a_2} + \boldsymbol{b_2} \dots\ \ \boldsymbol{a_n} + \boldsymbol{b_n}]$

So:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 6 & 7 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 9 \\ 2 & 3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 12 \\ 2 & 9 & 14 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} -1 & -2 \\ -3 & -4 \\ -5 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Transpose of a matrix A^T : interchange rows and column of A. If A is $m \times n$, then A^T is $n \times m$

$$\underbrace{\begin{bmatrix} 1 & 2 \\ \mathbf{3} & 4 \\ 5 & 6 \end{bmatrix}^T}_{[\ A\ ]^T} = \underbrace{\begin{bmatrix} 1 & \mathbf{3} & 5 \\ 2 & 4 & 6 \end{bmatrix}}_{A^T} \qquad \begin{bmatrix} 1 \\ \mathbf{2} \\ 3 \end{bmatrix}^T = [1 \quad \mathbf{2} \quad 3]$$

"**3**" $=$ the $\mathbf{(2,1)}$ **entry in A** is **the $(1,2)$ entry in A^T**

The (i,j) entry in A is the (j,i) entry in A^T

Properties of Matrix Addition/Scalar Multiplication & Transposes

(*all are obvious*)

If A, B, C are same size matrices and r, s are scalars:

$$A + B = B + A$$

$$A + 0 = A$$

$$A + (B + C) = (A + B) + C$$

$$r(A + B) = rA + rB$$

$$(r + s)A = rs + sA$$

$$r(sA) = (rs)A$$

$$(A^T)^T = A$$

$$(A + B)^T = A^T + B^T$$

$$(rA)^T = rA^T$$

Example of Multiplication of Matrices (see text or notes for the definition)

$$\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{2} & \mathbf{-1} \\ \mathbf{3} & \mathbf{4} \end{bmatrix} \begin{bmatrix} 0 & \mathbf{2} & 1 & 2 \\ -1 & \mathbf{2} & 0 & 2 \end{bmatrix}$$

$$= [\begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{2} & \mathbf{-1} \\ \mathbf{3} & \mathbf{4} \end{bmatrix} \begin{bmatrix} \mathbf{2} \\ \mathbf{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix}]$$

$$= [\begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \mathbf{2} \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{bmatrix} + \mathbf{2} \begin{bmatrix} \mathbf{0} \\ \mathbf{-1} \\ \mathbf{4} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix}]$$

$$= [\begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \begin{bmatrix} \mathbf{2(1) + 2(0} \\ \mathbf{2(2) + 2(-1)} \\ \mathbf{2(3) + 2(4)} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix}]$$

$$= \begin{bmatrix} 0 & \mathbf{2} & 1 & 2 \\ 1 & \mathbf{2} & 2 & 2 \\ -4 & \mathbf{14} & 3 & 14 \end{bmatrix}$$

Again, from the top:

$$C = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{2} & \mathbf{-1} \\ \mathbf{3} & \mathbf{4} \end{bmatrix} \begin{bmatrix} 0 & \mathbf{2} & 1 & 2 \\ -1 & \mathbf{2} & 0 & 2 \end{bmatrix} = \ldots$$

$$= \begin{bmatrix} 0 & \mathbf{2} & 1 & 2 \\ 1 & \mathbf{2} & 2 & 2 \\ -4 & \mathbf{2} & 3 & 14 \end{bmatrix}$$

$$\uparrow$$

$$\boldsymbol{c_{32} = 14 = a_{31}b_{12} + a_{32}b_{22}}$$

This illustrates the **Row-Column Rule for Computing AB**

If $AB = C = \begin{bmatrix} . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ & & \vdots & & \\ . & . & \boldsymbol{c_{ij}} & \cdots & . \\ . & . & . & \cdots & . \end{bmatrix}$

c_{ij} = entry in ith row, j th column of $C = AB$

= sum of products formed from

(<u>entries in i^{th} row of A</u>) $\cdot$ (corresponding entries in j^{th} column of B)

$$\boldsymbol{c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}}$$

Properties of Multiplication of Matrices

Assuming $A, B, C,$ all have the right size to be added or multiplied together; I_n is the $n \times n$ identity matrix.,

$$A(BC) = (AB)C$$

$$A(B + C) = AB + AC$$

$$(B + C)A = BA + CA$$

$$r(AB) = (rA)B = A(rB)$$

$$I_m A = A = A_n$$ (when A is $m \times n$)