**Columns of $C$ are**

**consumption <u>by</u> each sector from other sectors to produce 1 unit of its own product (measured in \$)**

$$\begin{array}{ccc} & & \begin{matrix} S_1 & \cdots & S_j & \cdots & S_n \\ & & & \downarrow & \end{matrix} \\ \textbf{From} \left\{ \begin{matrix} S_1 \\ \vdots \\ S_i \\ \vdots \\ S_n \end{matrix} \right. & \begin{matrix} \rightarrow \\ \\ \rightarrow \\ \\ \rightarrow \end{matrix} & \begin{bmatrix} c_{11} \cdots & c_{1j} & \cdots & c_{1n} \\ & \vdots & & \\ c_{i1} \cdots & c_{ij} & \cdots & c_{in} \\ & \vdots & & \\ c_{n1} \cdots & c_{nj} & \cdots & c_{nn} \end{bmatrix} \end{array}$$

$$\text{Column } j \quad \begin{bmatrix} c_{1j} \\ \vdots \\ c_{ij} \\ \vdots \\ c_{nj} \end{bmatrix} \text{ represents } ??$$

$$\text{Scalar multiple } x_j \begin{bmatrix} c_{1j} \\ \vdots \\ c_{ij} \\ \vdots \\ c_{nj} \end{bmatrix} = \begin{bmatrix} x_j c_{1j} \\ \vdots \\ x_j c_{ij} \\ \vdots \\ x_j c_{nj} \end{bmatrix} \text{ represents } ??$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \text{production vector: } x_1 \text{ units from } S_1, \ldots$$

What does product $C\boldsymbol{x}$ mean ?

$$C\boldsymbol{x} = \; x_1 \begin{bmatrix} c_{11} \\ \vdots \\ c_{\boldsymbol{i}1} \\ \vdots \\ c_{n1} \end{bmatrix} + x_2 \begin{bmatrix} c_{12} \\ \vdots \\ c_{i2} \\ \vdots \\ c_{n2} \end{bmatrix} + \ldots + x_n \begin{bmatrix} c_{1n} \\ \vdots \\ c_{\boldsymbol{in}} \\ \vdots \\ c_{nn} \end{bmatrix}$$

**from $\mathbf{S_1}$ ↘**

$$= \begin{bmatrix} \boldsymbol{c_{11}x_1 + \ldots + c_{1n}x_n} \\ \vdots \\ c_{i1}x_1 + \ldots + c_{in}x_n \\ \vdots \\ c_{n1}x_1 + \ldots + c_{nn}x_n \end{bmatrix}$$

$$= \begin{bmatrix} \textit{\textbf{total demanded \underline{from} sector }} \boldsymbol{S_1} \textit{\textbf{ to produce }} \boldsymbol{x} \\ \vdots \\ \textit{total demanded \underline{from} sector } S_2 \textit{ to produce } \boldsymbol{x} \\ \vdots \\ \textit{total demanded \underline{from} sector } S_n \textit{ to produce } \boldsymbol{x} \end{bmatrix}$$

What does a solution for the equation $C\boldsymbol{x} = \boldsymbol{x}$ mean ?

Add an <u>open</u> ("unproductive") <u>sector</u>: only consumes

$$\boldsymbol{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \text{demand from open sector}$$

***<u>Leontieff Open Economy Production Model</u>***

$$\boldsymbol{x} = C\boldsymbol{x} + \boldsymbol{d}$$

| | | | | |
|---|---|---|---|---|
| ***total production*** | = | ***demand from productive sector to produce $\boldsymbol{x}$*** | + | ***demand from open sector*** |
| | | *("intermediate demand")* | + | *("final demand")* |

Example

$$C = \begin{bmatrix} .10 & .05 & .30 & .20 \\ .15 & .25 & .05 & .10 \\ .30 & .10 & .10 & .25 \\ .15 & .20 & .10 & .20 \end{bmatrix}$$

$$(I - C) = \begin{bmatrix} .90 & -.05 & -.30 & -.20 \\ -.15 & .75 & -.05 & -.10 \\ -.30 & -.10 & .90 & -.25 \\ -.15 & -.20 & -.10 & .80 \end{bmatrix}$$

Suppose final demand $\boldsymbol{d} = \begin{bmatrix} 25000 \\ 10000 \\ 30000 \\ 50000 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} .90 & -.05 & -.30 & -.20 \\ -.15 & .75 & -.05 & -.10 \\ -.30 & -.10 & .90 & -.25 \\ -.15 & -.20 & -.10 & .80 \end{bmatrix}}_{(I-C)} \boldsymbol{x} = \underbrace{\begin{bmatrix} 25000 \\ 10000 \\ 30000 \\ 50000 \end{bmatrix}}_{\boldsymbol{d}}$$

Row reduce (Matlab)

$$\begin{bmatrix} .90 & -.05 & -.30 & -.20 & 25000 \\ -.15 & .75 & -.05 & -.10 & 10000 \\ -.30 & -.10 & .90 & -.25 & 30000 \\ -.15 & -.20 & -.10 & .80 & 50000 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 85580 \\ 0 & 1 & 0 & 0 & 50620 \\ 0 & 0 & 1 & 0 & 96160 \\ 0 & 0 & 0 & 1 & 103220 \end{bmatrix}$$

$\uparrow$

rounded to nearest \$

so $\boldsymbol{x} = C\boldsymbol{x} + \boldsymbol{d}$ if $\boldsymbol{x} = \begin{bmatrix} 85580 \\ 50620 \\ 96160 \\ 103220 \end{bmatrix}$, that is,

$S_1$ produces 85580 units (\$) ,
$S_2$ produces 50620 units (\$) , ...

## ***Leontieff Open Economy Production Model***

| | | | | |
|---|---|---|---|---|
| $\boldsymbol{x}$ | $=$ | $C\boldsymbol{x}$ | $+$ | $\boldsymbol{d}$ |
| ***total production*** | $=$ | ***demand from productive sector to produce $\boldsymbol{x}$*** | $+$ | ***demand from open sector*** |
| | | (*"intermediate demand"*) | $+$ | (*"final demand"*) |

Theorem about this model:

**If $C$, $\boldsymbol{d}$ have nonnegative entries and all column sums are $< 1$ (*every sector profitable*), then**

**$(I - C)$ must be invertible**

*and so $(I - C)\boldsymbol{x} = \boldsymbol{d}$ will have a unique solution: $\boldsymbol{x} = (I - C)^{-1}\boldsymbol{d}$*

**and all entries in solution $\boldsymbol{x}$ will be $\geq 0$**

*and so the solution is economically feasible*

For a $2 \times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the <u>determinant</u>

<u>of $A$</u> is the number $\boldsymbol{ad - bc}$

If **det($A$)** $\neq 0$, then $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible and

$$A^{-1} = \begin{bmatrix} \frac{d}{(ad-bc)} & -\frac{b}{(ad-bc)} \\ -\frac{c}{(ad-bc)} & \frac{a}{(ad-bc)} \end{bmatrix}$$

because it works:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \frac{d}{(ad-bc)} & -\frac{b}{(ad-bc)} \\ -\frac{c}{(ad-bc)} & \frac{a}{(ad-bc)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{ad}{(ad-bc)} - \frac{bc}{(ad-bc)} & \frac{-ab}{(ad-bc)} + \frac{ab}{(ad-bc)} \\ \frac{cd}{(ad-bc)} - \frac{dc}{(ad-bc)} & \frac{-cb}{(ad-bc)} + \frac{ad}{(ad-bc)} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \; = \mathbf{a_{11} \, det} \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - \mathbf{a_{12} \, det} \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$$

$$+ \mathbf{a_{13} \, det} \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$\begin{aligned} = \quad & a_{11}(a_{22}a_{33} - a_{23}a_{32}) \\ & - a_{12}(a_{21}a_{33} - a_{23}a_{31}) \\ & + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned}$$

$$\begin{aligned} = \quad & a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ & - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \end{aligned}$$

Motivation behind the definition of determinant, using an invertible $3 \times 3$ matrix $A$

Assume $a_{11} \neq 0$ (or do row interchanges to make this true)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11}a_{21} & a_{11}a_{22} & a_{11}a_{23} \\ a_{11}a_{31} & a_{11}a_{32} & a_{11}a_{33} \end{bmatrix}$$

$$\sim \begin{bmatrix} a_{11} \neq 0 & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31} \end{bmatrix}$$

(assume $a_{11}a_{22} - a_{12}a_{21} \neq 0$,
or else swap row 2 and row 3 to make it true)

$\vdots$ etc

$$\begin{bmatrix} a_{11} \neq 0 & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} \neq 0 & a_{11}a_{23} - a_{13}a_{21} \\ 0 & 0 & a_{11}\Delta \end{bmatrix}$$

$\nearrow$

$$\Delta = ( \; \boldsymbol{a_{11}a_{22}a_{33}} + \boldsymbol{a_{12}a_{23}a_{31}} + \boldsymbol{a_{13}a_{21}a_{32}} - \boldsymbol{a_{11}a_{23}a_{32}} - \boldsymbol{a_{12}a_{21}a_{33}} - \boldsymbol{a_{13}a_{22}a_{31}} \; )$$

So if $A$ is invertible, we must have $\Delta \neq 0$

$A$ is invertible means that det $A = \Delta \neq 0$

($\Delta$ is really the same formula as we used to define det $A$)

So (for $3 \times 3$ matrices) the calculation illustrates why we defined det $A$ the way we did

$$A \text{ is invertible } \Rightarrow \det A \neq 0 \text{ (for the } 3 \times 3 \text{ case)}$$

Will see that's true when $A$ is $n \times n$

$A = n \times n$ matrix

$A_{ij} = (n-1 \times (n-1)$ matrix remaining after

erasing $i^{th}$ row and $j^{th}$ column of $A$

$C_{ij} = (i, j)$ cofactor of $A =$ the number

$= (-1)^{i+j} \det(A_{ij})$

Definition of det $A$ was (going across first row of $A$)

$$\mathbf{det}\boldsymbol{A} = \boldsymbol{a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{1n}C_{1n}}$$

FACT: any row or column can be used to compute det $A$

**across $\boldsymbol{i^{th}}$ row:**

$$\mathbf{det}\,\boldsymbol{A} = \boldsymbol{a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}}$$

or

**down $\boldsymbol{j^{th}}$ column:**

$$\mathbf{det}\,\boldsymbol{A} = \boldsymbol{a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}}$$

**Example** det $\begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix}$

Cofactor expansion down the third column:

$$a_{13}\, C_{13} \qquad\qquad + a_{23}C_{23} \quad + a_{33}C_{33}$$

$$= a_{13}(-1)^{1+3}\det A_{13} + a_{23}(-1)^{2+3}\det A_{23}$$

$$+ a_{33}(-1)^{3+3}\det A_{33}$$

$$= 3\det\begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix} - 0\det\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} + 1\det\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

$$= 3(-2) + 0 + 1(2) = -4$$

Easiest to choose a row or column with lots of 0's (across 2nd row)

$$-0\det\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} + 2\det\begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$$

$$-0\det\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = 2(-2) = -4$$

**Theorem** Let $A$ be a square matrix:

1) if a multiple of one row of $A$ is added to another to get a matrix $B$, then $\det A = \det B$

2) If two rows of $A$ are interchanged to get $B$, then $\det B = -\det A$

3) If one row of $A$ is multiplied by $k$ to get $B$, then $\det B = k \det A$

Note re 3) : $\det \underset{\substack{\uparrow \\ B}}{\begin{bmatrix} ka & kb \\ c & d \end{bmatrix}} = k \det \underset{\substack{\uparrow \\ A}}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}$

*3) is often used this way to "factor out k" from a row*

$$\det\begin{bmatrix}1 & 1 & 3\\ 0 & 2 & 0\\ 1 & 2 & 1\end{bmatrix} = -\det\begin{bmatrix}1 & 1 & 3\\ 1 & 2 & 1\\ 0 & 2 & 0\end{bmatrix}$$

$$= -\det\begin{bmatrix}1 & 1 & 3\\ 0 & 1 & -2\\ 0 & 2 & 0\end{bmatrix}$$

$$= -\det\begin{bmatrix}1 & 1 & 3\\ 0 & 1 & -2\\ 0 & 0 & 4\end{bmatrix} = -4$$

**A side observation, for $3 \times 3$ matrix $A$ :**

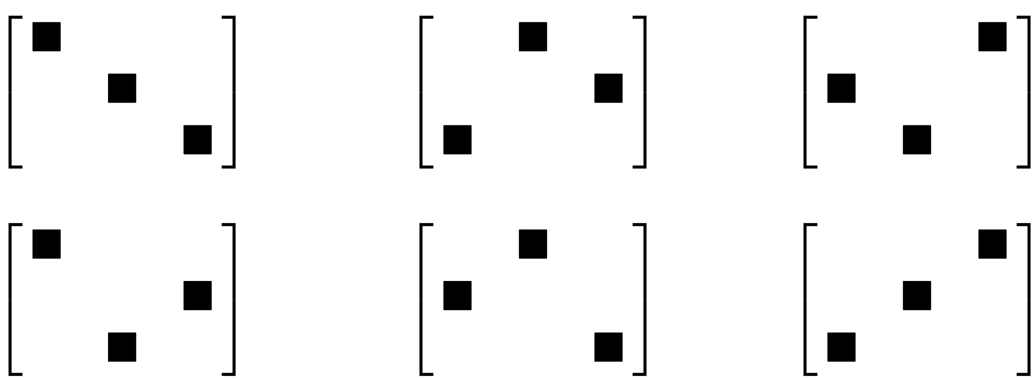
$$\det A = \Delta =$$

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

$\Delta$ is the sum of 6 terms, each with a $+$ or $-$ attached. Each term is

- a product of 3 entries from $A$: no two from the same row, no two from the same column and

- every "triple" of that kind is included



- the sign on a term is determined looking at its sequence of "column subscripts" and seeing whether an odd ( $-$ ) or even ( $+$ ) number of "adjacency switches" to get this sequence into natural order (1,2,3)

$$
\begin{aligned}
\text{e.g.}\quad & +a_{12}a_{23}a_{31}: \ (2,3,1) \rightarrow (2,1,3) \rightarrow (1,2,3) \\
& -\ a_{12}a_{21}a_{33}: \ (2,1,3) \rightarrow (1,2,3)
\end{aligned}
$$