Matrix Terminology & Operations

$$[\ \boldsymbol{a_1} \quad \boldsymbol{a_2} \quad \dots \quad \boldsymbol{a_j} \quad \dots \quad \boldsymbol{a_n}\ ]$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ & & \vdots & & & \\ a_{i1} & a_{i2} & \dots & \boldsymbol{a_{ij}} & \dots & a_{in} \\ & & \vdots & & & \\ a_{m1} & a_{n2} & \dots & a_{1j} & \dots & a_{mn} \end{bmatrix}$$

$\boldsymbol{a_{ij}}$ = element of $\boldsymbol{A}$ in $\boldsymbol{i^{th}}$ row and $\boldsymbol{j^{th}}$ column

The entries a_{ii} form the main diagonal of A, for example:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \\ a_{51} & a_{52} & a_{53} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

A diagonal matrix: square and $a_{ij} = 0$ when $i \neq j$

Examples: $\begin{bmatrix} \mathbf{6} & 0 & 0 \\ 0 & -\boldsymbol{\pi} & 0 \\ 0 & 0 & \mathbf{\frac{13}{7}} \end{bmatrix}$ $\begin{bmatrix} \mathbf{0} & 0 \\ 0 & \mathbf{0} \end{bmatrix}$

Identity matrix I_n : the $n \times n$ diagonal matrix that has only 1's on the diagonal; it has columns $\boldsymbol{e_1}, \ldots, \boldsymbol{e_n}$

Example : $I_3 = \begin{bmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \end{bmatrix}$

$\uparrow$ $\uparrow$ $\uparrow$

$\boldsymbol{e_1}$ $\boldsymbol{e_2}$ $\boldsymbol{e_3}$

A zero matrix : all $a_{ij} = 0$

Example: $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Operations with Matrices

Scalar times a matrix: defined by "multiply each column by the scalar"

$$
\begin{aligned}
A &= [\boldsymbol{a}_1 \quad \boldsymbol{a}_2 \ \dots \ \boldsymbol{a}_n] \\
cA &= c\,[\boldsymbol{a}_1 \quad \boldsymbol{a}_2 \ \dots \ \boldsymbol{a}_n] \\
&= [c\boldsymbol{a}_1 \ c\boldsymbol{a}_2 \dots c\boldsymbol{a}_n]
\end{aligned}
$$

Example: $$3\begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & 6 \\ 2 & 1 & 7 \end{bmatrix} = \begin{bmatrix} 3 & 6 & -1 \\ 0 & 15 & 18 \\ 6 & 3 & 21 \end{bmatrix}$$

Sum of two matrices: defined by "add the columns"
(matrices must be same size $m \times n$)

$$
\begin{aligned}
&A = [\boldsymbol{a}_1 \ \boldsymbol{a}_2 \dots \boldsymbol{a}_n], \qquad B = [\boldsymbol{b}_1 \ \boldsymbol{b}_2 \dots \boldsymbol{b}_n] \\
&A + B = [\boldsymbol{a}_1 + \boldsymbol{b}_1 \quad \boldsymbol{a}_2 + \boldsymbol{b}_2 \ \dots \ \boldsymbol{a}_n + \boldsymbol{b}_n]
\end{aligned}
$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 6 & 7 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 9 \\ 2 & 3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 12 \\ 2 & 9 & 14 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} -1 & -2 \\ -3 & -4 \\ -5 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Properties of Matrix Addition/Scalar Multiplication & Transposes

All these properties are obvious

If A, B, C are same size matrices and r, s are scalars:

$A + B = B + A$ because

$$\begin{array}{ccccc} A + B = [\, \boldsymbol{a}_1 + \boldsymbol{b}_1 & \boldsymbol{a}_2 + \boldsymbol{b}_2 & \ldots & \boldsymbol{a}_n + \boldsymbol{b}_n] \\ \| & \| & & \| \\ B + A = [\, \boldsymbol{b}_1 + \boldsymbol{a}_1 & \boldsymbol{b}_2 + \boldsymbol{a}_2 & \ldots & \boldsymbol{b}_n + \boldsymbol{a}_n] \end{array}$$

$A + 0 = A$ (where $0 =$ the zero matrix with same size as A)

$A + (B + C) = (A + B) + C$

$r(A + B) = rA + rB$

$(r + s)A = rA + sA$

$r(sA) = (rs)A$

Transpose of a matrix A^T : interchange rows and columns of A. If A is $m \times n$, then A^T is $n \times m$

$$\underbrace{\begin{bmatrix} 1 & 2 \\ \mathbf{3} & 4 \\ 5 & 6 \end{bmatrix}^T}_{\uparrow \\ [\ A\ ]^T} = \underbrace{\begin{bmatrix} 1 & \mathbf{3} & 5 \\ 2 & 4 & 6 \end{bmatrix}}_{\uparrow \\ \boldsymbol{A^T}} \quad \text{and} \quad \begin{bmatrix} 1 \\ \mathbf{3} \\ 2 \end{bmatrix}^T = \begin{bmatrix} 1 & \mathbf{3} & 2 \end{bmatrix}$$

Because rows and columns are interchanged,

the $(2, 1)$ entry in A is **the $(1, 2)$ entry in A^T**

the (i, j) entry in A^T is the (j, i) entry in A

Simple Properties of Transposes

$$(A^T)^T = A$$

$$(A + B)^T = A^T + B^T$$

Add $A + B$, then transpose

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 4 & 4 & 4 \end{bmatrix}$$

$$\xrightarrow{T} \begin{bmatrix} 1 & 4 \\ 1 & 4 \\ 2 & 4 \end{bmatrix}$$

Transpose each of A, B then add

$$||$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 1 & 4 \\ 2 & 4 \end{bmatrix}$$

$$(rA)^T = rA^T$$

Product of Matrices:

$$\begin{array}{c} \\ \\ \boldsymbol{AB} = \end{array} \begin{array}{c} \\ \\ \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{2} & -\mathbf{1} \\ \mathbf{3} & \mathbf{4} \end{bmatrix} \\ \boldsymbol{A} \\ \uparrow \\ 3 \times \mathbf{2} \end{array} \begin{array}{c} \boldsymbol{b_1} \;\; \boldsymbol{b_2} \;\; \boldsymbol{b_3} \;\; \boldsymbol{b_4} \\ \downarrow \;\; \downarrow \;\; \downarrow \;\; \downarrow \\ \begin{bmatrix} 0 & 2 & 1 & 2 \\ -1 & 2 & 0 & 2 \end{bmatrix} \\ \boldsymbol{B} \\ \uparrow \\ \mathbf{2} \times 4 \end{array} \begin{array}{c} \\ \\ \text{means ???} \end{array}$$

Each column $\boldsymbol{b_1}$, $\boldsymbol{b_2}$, $\boldsymbol{b_3}$, $\boldsymbol{b_4}$ is in $\mathbb{R}^2$ so $A\boldsymbol{b_1}$, $A\boldsymbol{b_2}$, $\boldsymbol{Ab_3}$, $A\boldsymbol{b_4}$ all make sense.

So $\begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 & 2 \\ -1 & 2 & 0 & 2 \end{bmatrix} =$

$$= \left[A\begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad A\begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad A\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad A\begin{bmatrix} 2 \\ 2 \end{bmatrix} \right]$$

$$= \left[A\begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{2} & \mathbf{-1} \\ \mathbf{3} & \mathbf{4} \end{bmatrix}\begin{bmatrix} \mathbf{2} \\ \mathbf{2} \end{bmatrix} \quad A\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad A\begin{bmatrix} 2 \\ 2 \end{bmatrix} \right]$$

$$\qquad \boldsymbol{Ab_1} \qquad\qquad \boldsymbol{Ab_2} \qquad\qquad\qquad \boldsymbol{Ab_3} \qquad \boldsymbol{Ab_4}$$

$$= \left[A\begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \mathbf{2}\begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{bmatrix} + \mathbf{2}\begin{bmatrix} \mathbf{0} \\ \mathbf{-1} \\ \mathbf{4} \end{bmatrix} \quad A\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad A\begin{bmatrix} 2 \\ 2 \end{bmatrix} \right]$$

$$= \left[A\begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \begin{bmatrix} \mathbf{2(1) + 2(0)} \\ \mathbf{2(2) + 2(-1)} \\ \mathbf{2(3) + 2(4)} \end{bmatrix} \quad A\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad A\begin{bmatrix} 2 \\ 2 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 0 & \mathbf{2} & 1 & 2 \\ 1 & \mathbf{2} & 2 & 2 \\ -4 & \mathbf{14} & 3 & 14 \end{bmatrix}$$

<u>Definition</u>: If A is $m \times n$ and B is $n \times p$,

$$B = [\, b_1 \ b_2 \ \cdots \ b_p]$$

<u>Define</u> $C = AB = [A\boldsymbol{b_1} \quad A\boldsymbol{b_2} \quad A\boldsymbol{b_n}]$

↑

Why??

$$C = AB = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{2} & -\mathbf{1} \\ \color{red}{\mathbf{3}} & \color{red}{\mathbf{4}} \end{bmatrix} \begin{bmatrix} 0 & \color{red}{\mathbf{2}} & 1 & 2 \\ -1 & \color{red}{\mathbf{2}} & 0 & 2 \end{bmatrix}$$

$$\underset{A}{} \qquad [\ \boldsymbol{b_1} \ \ \boldsymbol{b_2} \ \ \boldsymbol{b_2} \ \ \boldsymbol{b_4}]$$

$$= \left[\underset{\boldsymbol{Ab_1}}{A\begin{bmatrix} 0 \\ -1 \end{bmatrix}} \quad \underset{\boldsymbol{Ab_2}}{\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{2} & -\mathbf{1} \\ \color{red}{\mathbf{3}} & \color{red}{\mathbf{4}} \end{bmatrix} \begin{bmatrix} \color{red}{\mathbf{2}} \\ \color{red}{\mathbf{2}} \end{bmatrix}} \quad \underset{\boldsymbol{Ab_3}}{A\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \quad \underset{\boldsymbol{Ab_4}}{A\begin{bmatrix} 2 \\ 2 \end{bmatrix}} \right]$$

$$= \left[A\begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \begin{bmatrix} \mathbf{2(1) + 2(0} \\ \mathbf{2(2) + 2(-1)} \\ \color{red}{\mathbf{3(2) + 4(2)}} \end{bmatrix} A\begin{bmatrix} 1 \\ 0 \end{bmatrix} \ A\begin{bmatrix} 2 \\ 2 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 0 & \color{blue}{\mathbf{2}} & 1 & 2 \\ 1 & \color{blue}{\mathbf{2}} & 2 & 2 \\ -4 & \color{red}{\mathbf{14}} & 3 & 14 \end{bmatrix} = AB$$

$$\uparrow$$

$$\color{red}{\boldsymbol{c_{32}} = \mathbf{14} = (\mathbf{Row_3}\boldsymbol{A})(\mathbf{Col_2}\boldsymbol{B})}$$

Row-Column Rule: $(\text{Row}_i A)(\text{Col}_j B) = \boldsymbol{c_{ij}}$

and

$(\text{Row}_i A)B = \text{Row}_i(AB)$

EXTRA

$$C = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ \mathbf{3} & \mathbf{4} \end{bmatrix} \begin{bmatrix} 0 & \mathbf{2} & 1 & 2 \\ -1 & \mathbf{2} & 0 & 2 \end{bmatrix} = \ldots$$

$$= \begin{bmatrix} 0 & 2 & 1 & 2 \\ 1 & 2 & 2 & 2 \\ -4 & \mathbf{14} & 3 & 14 \end{bmatrix}$$

$$\boldsymbol{c_{32} = 14 = a_{31}b_{12} + a_{32}b_{22}}$$

Row-Column Rule for Computing AB

c_{ij} = entry in ith row, j th column of AB

= sum of products (entries in i^{th} row of A)

$\cdot$ (corresponding entries in j^{th} column of B)

$$\begin{bmatrix} a_{i1} & a_{i2} & \ldots & a_{in} \end{bmatrix} \cdot \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ \vdots \\ b_{nj} \end{bmatrix}$$

$$\boldsymbol{c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}}$$

$$\boldsymbol{c_{ij} = \mathbf{Row}_i(A) \cdot \mathbf{Col}_j(B)}$$

Even further, notice

$$\begin{bmatrix} & & \vdots & & \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ & & \vdots & & \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1p} \\ \vdots & \vdots & & & \vdots & \vdots \\ b_{j1} & b_{j2} & \cdots & b_{jj} & \cdots & b_{jp} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix}$$

$$= \begin{bmatrix} & & & & & \\ c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{in} \\ & & & & & \end{bmatrix}$$

$$\uparrow$$

$$c_{ij} = \textbf{Row}_i(A) \cdot \textbf{Col}_j(B)$$

For the whole i^{th} row in the product:

$$\underset{(1 \times p)}{\textbf{row}_i(AB)} = \underset{(1 \times n)}{\textbf{row}_i(A)} \cdot \underset{(n \times p)}{B}$$

Properties of Multiplication of Matrices

Assuming $A, B, C,$ are all the right sizes to be added or multiplied together and that I_n is the $n \times n$ identity matrix,

$$A(BC) = (AB)C$$

$$A(B + C) = AB + AC$$

$$(B + C)A = BA + CA$$

$$r(AB) = (rA)B = A(rB)$$

$I_m A = A = A_n$ (when A is $m \times n$)