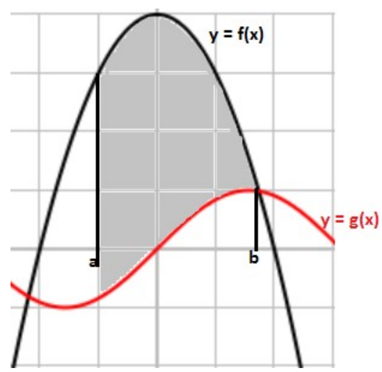
Lecture 5, January 27, 2017

<u>Review example</u>: $\int \frac{\cos(5x)}{e^{\sin(5x)}} dx$ (substitute $u = \sin 5x$; then $\frac{1}{5}du = \cos 5x\, dx$)

$$= \frac{1}{5}\int \frac{1}{e^u}\, du = \frac{1}{5}\int e^{-u}\, du = \text{(substitute } z = -u\text{, so } -dz = du)$$

$$= -\frac{1}{5}\int e^z dz = -\frac{1}{5}e^z + C = -\frac{1}{5}e^{-u} + C = -\frac{1}{5}e^{-\sin(5x)} + C$$

From last time: a region (*shaded*) lying between $x = a$ and $x = b$, with top boundary curve $y = f(x)$ and bottom boundary curve $y = g(x)$ has area $\int_a^b f(x) - g(x)\ dx$



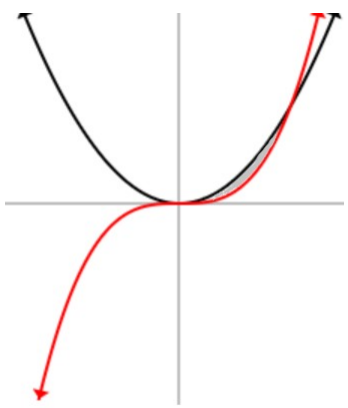
Q1: Draw a picture and then write an integral that gives the area between the graphs of $y = x^2$ and $y = x^3$

A) $\int_0^2 x^2 - x^3\,dx$ B) $\int_0^1 x^3 - x^2\,dx$ C) $\int_0^1 (\sqrt{x} - \sqrt[3]{x}\,)\,dx$

D) $\int_0^1 x^2 - x^3\,dx$ E) $\int_0^1 \sqrt[3]{x} - \sqrt{x}\,dx$

Answer

The curves intersect where $x^3 - x^2 = 0$, so, where $x = 0$ and $x = 1$
Between 0 and 1, $x^3 < x^2$ (for example, $(\frac{1}{2})^3 < (\frac{1}{2})^2$

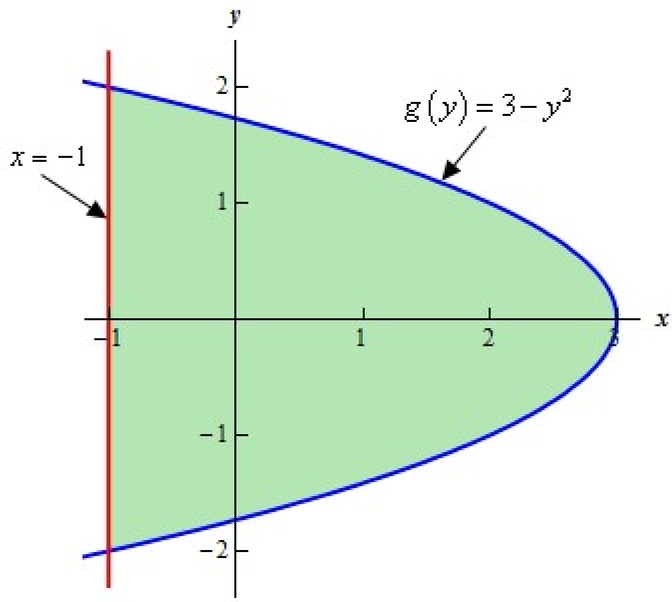


The shaded area between the curves is

$$\int_0^1 (\text{top boundary function} - \text{bottom boundary function}) = \int_0^1 x^2 - x^3\,dx$$

(*The value is* $\frac{x^3}{3} - \frac{x^4}{4}\Big|_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$.)

Example: Find the shaded region



The region lies between $x = -1$ and $x = 3$. The functions that give the top and bottom boundaries are <u>not given</u>, but we can figure them out:

We have the curve $x = g(x) = 3 - y^2$, so $\quad y^2 = 3 - x$

$$y = \pm\sqrt{3 - x}$$

The upper boundary of the region is the graph of the function $y = f(x) = \sqrt{3 - x}$
The lower boundary of the region is the graph of the function $y = g(x) = -\sqrt{3 - x}$

The earlier formula then gives

$$\text{Shaded area} = \int_{-1}^{3} (\sqrt{3 - x}) - (-\sqrt{3 - x})\, dx = \int_{-1}^{3} 2\sqrt{3 - x}\, dx$$

$$= \text{(substitute } u = 3 - x\text{)} \quad \int_{4}^{0} -2\sqrt{u}\, du$$

$$= \textit{(for convenience)} \quad \int_{0}^{4} 2\sqrt{u}\, du = \frac{4}{3} u^{3/2} \Big|_{0}^{4} = \frac{32}{3}$$

We can also use the formula (*<u>see textbook/class notes</u>*) describing the area in terms of right and left boundary curves" $x = f(y) = 3 - y^2$ and $x = g(y) = -1$.

Form the equation $x = 3 - y^2$, the upper and lower left corner points of the region are $(-1, \pm 2)$. The area is given by

$\int_{-2}^{2}$(right boundary curve $-$ left boundary cruve) dy

$= \int_{-2}^{2}(3-y^2)-(-1)\,dy = \int_{-2}^{2} 4-y^2\,dy$

$= 2\int_0^2 4-y^2\,dy = 2(4y-\frac{y^3}{3})\Big|_0^2 = 2(8-\frac{8}{3}) = \frac{32}{3}$

$\uparrow$

optional maneuver to
make calcuoations easier: ok because
$h(y)=4-y^2$ *is a even function: see last lecture*

Q2: Write a different integral that gives the area between the graphs of $y=x^2$ and $y=x^3$.

A) $\int_0^2 y^2-y^3\,dy$ B) $\int_0^1 y^3-y^2\,dy$ C) $\int_0^1 (\sqrt{y}-\sqrt[3]{y}\,)\,dy$
D) $\int_0^1 y^2-y^3\,dy$ E) $\int_0^1 \sqrt[3]{y}-\sqrt{y}\;dy$

(Refer to Q 1, same drawing.) Written in terms of y, the boundary curves are

right boundary: $y=x^3 \longrightarrow x=\sqrt[3]{y}$ and
left boundary: $y=x^2 \longrightarrow x=\sqrt{y}$

The region lies between $y=0$ and $y=1$. The area is

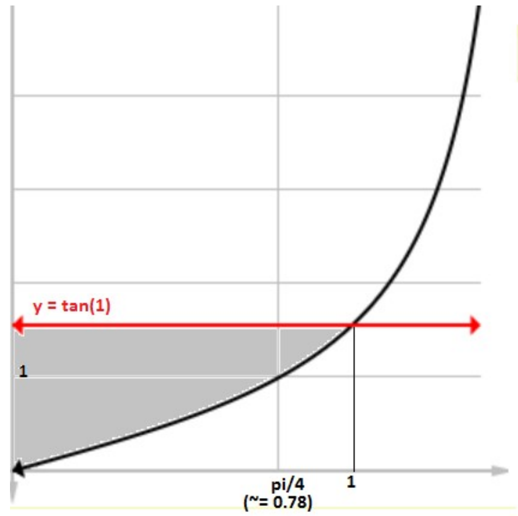
$\int_0^1$(right boundary curve $-$ left boundary curve) dy

$= \int_0^1 \sqrt[3]{y}-\sqrt{y}\,dy = \int_0^1 y^{1/3}-y^{1/2}\,dy = \frac{3}{4}y^{4/3}-\frac{2}{3}y^{3/2}\Big|_0^1 =$

$= \frac{3}{4}-\frac{2}{3}=\frac{1}{12}$ *(same answer, of course, as in Q1)*

<u>Example</u> Find the area of the region bounded by $y=\tan x$, $y=\tan(1)$ (*a horizontal line!*), and $x=0$ (*a vertical line!*)

The picture looks like



Method I: the shaded area is $\int_0^1 \tan(1) - \tan(x)\,dx$

$$= \left(\tan(1)\cdot x - \ln|\sec x| \,\right)\Big|_0^1$$

$$= [\tan(1)\cdot 1 - \ln(\sec 1)] - [\tan(1)\cdot 0 - \ln(\sec 0)]$$

$$= [\tan(1)\cdot 1 - \ln(\sec 1)] - [\tan(1)\cdot 0 - \ln(1)]$$

$$= [\tan(1)\cdot 1 - \ln(\sec 1)] - [\tan(1)\cdot 0 - \ln(\sec 0)]$$

$$= \tan(1)\cdot 1 - \ln(\sec 1) \qquad (\approx 0.9418)$$

Method II: The region lies between $y = 0$ and $y = \tan(1)$; the left boundary is $x = 0$ and the right boundary is $x = \arctan y$ (from solving $y = \tan x$ for x), so

the shaded area is $\int_0^{\tan(1)} \arctan y - 0\,dy = \int_0^{\tan(1)} \arctan y\,dy$

We have not yet learned any way to find the antiderivative $\int \arctan y\,dy$. Therefore only Method I works for us right now. However, in fact,

$\int \arctan y\,dy = y\arctan y - \frac{1}{2}\ln(1+y^2)$ *(which you <u>can</u> verify by differentiating!)*

Given that fact:

$$\text{Area} = \int_0^{\tan(1)} \arctan y\,dy = \left(y\arctan y - \tfrac{1}{2}\ln(1+y^2) \,\right)\Big|_0^{\tan(1)}$$

$$= \tan(1)\arctan(\tan 1) - \tfrac{1}{2}\ln(1+\tan^2(1)) - (0 - \tfrac{1}{2}\ln(1))$$

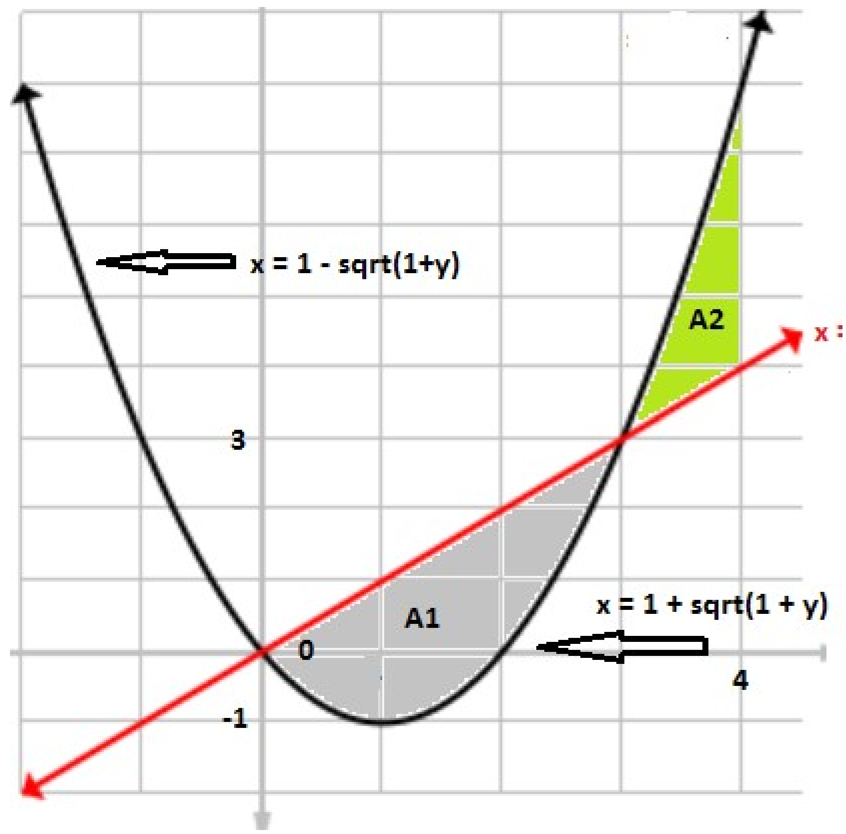
$$= \tan(1) \cdot 1 - \tfrac{1}{2}\ln(1 + \tan^2(1))$$

$$= \tan(1) - \tfrac{1}{2}\ln\left(\sec^2 1\right) = \tan(1) - \ln\left((\sec^2 1)^{1/2}\right)$$

$$= \tan(1) - \ln\left(\sec 1\right) \qquad \text{(\textit{same answer as Method I!})}$$

Extra Example, not used in either lecture

This same picture was used in Lecture 4: The areas were computed in terms of x. (*See Lecture 4 notes*). Here, the boundary curves are written as functions of y.



For the region A1: the part of A1 below the x axis has

right boundary $x = 1 + \sqrt{1+y}$ and
left boundary $x = 1 + \sqrt{1+y}$

Then we think of "right boundary curve $-$ left boundary curve as giving the "height" of a rectangular strip (lying sideways) with width Δy. This leads to the integral

$$\int_{-1}^{0}(1+\sqrt{1+y}\,) - (1-\sqrt{1+y}\,)\,dy = \int_{-1}^{0} 2\sqrt{1+y}\,dy$$

(substitute $u = 1 + y$)

$$= \int_{0}^{1} 2\sqrt{u}\,du = \tfrac{4}{3}u^{3/2}\big|_0^1 = \tfrac{4}{3}$$

The area for the part of A1 that lies above the x-axis is

$$\int_{0}^{3}(1+\sqrt{1+y} - y\ dy \ = \int_{0}^{3} 1+\sqrt{1+y}\,dy - \int_{0}^{3} y\ dy$$

(substitute $u = y + 1$ in the first integral)

$$= \int_0^3 1 + \sqrt{1+y}\, dy \; - \int_0^3 y \; dy = \int_1^4 (1 + \sqrt{u}\,)\, du - \int_0^3 y \; dy$$

$$= (y + \tfrac{2}{3} y^{3/2})\Big|_1^4 - \tfrac{y^2}{2}\Big|_0^3 = (4 + \tfrac{16}{3} - 1 - \tfrac{2}{3}) - (\tfrac{9}{2} - 0)$$

$$= \tfrac{24 + 32 - 6 - 4 - 27}{6} = \tfrac{19}{6}$$

Putting the 2 areas together, we'd get area A1 $= \frac{4}{3} + \frac{19}{6} = \frac{27}{6} = \frac{9}{2}$ (of course, the same as the answer for area A1 obtained in Lecture 4).