Sums of Powers of Natural Numbers

Let's use the symbol S_k for the sum of the k^{th} powers of the first n natural numbers. In other words,

$$S_k = 1^k + 2^k + \ ... \ + n^k.$$

Of course, this is a "formula" for S_k, but it doesn't help you compute $-$ it doesn't tell you how to find the exact value, say, of $S_3 = 1^3 + 2^3 + ... + 15^3$. We'd like to get what's called a closed formula for S_k $-$ meaning one without the annoying " ... " in it.

For $S_0 = 1^0 + 2^0 + \ ... \ + n^0$, this is easy: since there are n terms, each equal to 1, so we get

$$S_0 = 1 + 1 + ... + 1 \ = 1 \cdot n \ = n$$

For S_1, it's already harder. Here's a neat way of finding a closed formula for S_1:

Write down the sum S_1 twice, in two different orders:

$$\begin{array}{lccccccccc} S_1 = & 1 & + & 2 & + & 3 & + ... + & (n-1) & + & n \\ S_1 = & n & + & (n-1) & + & (n-2) & + ... + & 2 & + & 1 \end{array}$$

Then add to get:

$$2S_1 = (n+1) \ + \ (n+1) + \ (n+1) + ... \ + \ (n+1) + (n+1).$$

Since $(n+1)$ appears n times on the right side,

$2S_1 = n(n+1)$, so

$$S_1 = \frac{n(n+1)}{2}$$

This is a "usable" closed formula: for example, $1 + 2 + 3 + ... + 15 = \frac{15(16)}{2} = 120$.

The argument above shows how somebody might actually "discover" the formula. (Of course, if someone just gave you this as a proposed formula, you could then verify that it always works using mathematical induction, a method you should have seen in a precalculus course.)

There are lots of such formulas. They often come up in Calculus I when integrals are introduced: they are useful for evaluating integrals like $\int_0^1 x\,dx$, $\int_0^1 x^2\,dx$, ... directly from the definition of the integral (without using the Fundamental Theorem of Calculus. (*For example, see pp. 368 in the textbook, Stewart, Calculus Early Transcendentals, 8th edition.*) Here's a list of some of those formulas. You should try proving one or more of them using induction.

$$\begin{aligned} S_0 &= 1^0 + 2^0 + \ ... \ + n^0 = n \\ S_1 &= 1^1 + 2^1 + ... \ + n^1 = \tfrac{n(n+1)}{2} \\ S_2 &= 1^2 + 2^2 + ... \ + n^2 = \tfrac{n(n+1)(2n+1)}{6} \\ S_3 &= [\tfrac{n(n+1)}{2}]^2 \quad \text{(Curious observation: } S_3 = [S_1]^2\text{)} \end{aligned}$$

Where do these formulas come from? Each one can be proved by induction if you are given the formula. But what, for example is S_4? Did somebody find the S_3 formula by looking at lots of values of n and then guessing a formula that would fit her calculations?

There is a way to get a formula for each S_k once you know the previous ones. This is called a recursive formula for S_k: one that's given in terms of the preceding formulas $S_0, S_1, \ldots, S_{k-1}$. Here's how it works (an idea I first read in George Polya's book, *Mathematical Discovery*):

We can see directly that $S_0 = n$. How can we use S_0 to find S_1?

For any positive integer j, we know that $(j+1)^2 - j^2 = 2j+1$. We write this down for each value $j = 1, 2, \ldots, n$

$$\begin{aligned}
2^2 - 1^2 &= 2(1)+1 \\
3^2 - 2^2 &= 2(2)+1 \\
4^2 - 3^2 &= 2(3)+1 \\
&\vdots \\
n^2 - \ldots & \\
(n+1)^2 - n^2 &= 2(n)+1
\end{aligned}$$

Adding up the columns on both sides (lots of cancellations happen on the left-hand side) gives

$$\begin{aligned}
(n+1)^2 - 1 &= 2(1+2+\ldots+n)+n \\
&= 2S_1 + n \quad \text{so} \\
n^2+2n+1-1 &= 2S_1 + n \quad \text{so} \\
n^2+n = 2S_1 & \quad \text{so} \\
\tfrac{n^2+n}{2} = \tfrac{n(n+1)}{2} &= S_1.
\end{aligned}$$

Now we know formulas for S_0 and S_1. How can we get a formula for S_2 ? It's exactly the same idea, but a little more algebra. For any j we know that $(j+1)^3 - j^3 = 3j^2+3j+1$. Write this down for each $j = 1, 2, \ldots, n$.

$$\begin{aligned}
2^3 - 1^3 &= 3(1^2)+3(1)+1 \\
3^3 - 2^3 &= 3(2^2)+3(2)+1 \\
4^3 - 3^3 &= 3(3^2)+3(3)+1 \\
\ldots & \\
n^3 - \ldots & \\
(n+1)^3 - n^3 &= 3(n^2)+3(n)+1.
\end{aligned}$$

Adding up the columns on both sides gives

$$\begin{aligned}
(n+1)^3 - 1 &= 3(1^2+2^2+\ldots+n^2) + 3(1+2+\ldots+n) + (1+1+\ldots+1) \\
&= 3\,S_2 + 3\,S_1 + S_0
\end{aligned}$$

so $n^3+3n^2+3n+1-1 = 3S_2+3S_1+S_0$.

Then we solve to get S_2 : $n^3+3n^2+3n-3S_1-S_0 = 3S_2$, so

$S_2 = \frac{n^3+3n^2+3n-3S_1-S_0}{3}$. This is a recursive formula for S_2.

If we like, since we know formulas for S_1 and S_0, we can substitute, simplify, and get

$$\begin{aligned}
S_2 &= \frac{n^3+3n^2+3n-3[\frac{n(n+1)}{2}]-n}{3} = \frac{\frac{2n^3+6n^2+6n-3n^2-3n-2n}{2}}{3} \\
&= \frac{2n^3+3n^2+n}{6} = \ldots = \frac{n(n+1)(2n+1)}{6}
\end{aligned}$$

See if you can now derive the formula for S_3 (mentioned above), making use of the algebraic identity $(j+1)^4 - j^4 = 4j^3+6j^2+4j+1$.

Some Additional Material

If you know the binomial formula (from high school) and can therefore expand $(j+1)^k$, then the same idea works for any natural number k. But the bigger k is, the more algebra is involved. An outline goes like this.

The formula for the "binomial coefficients" : $\binom{k}{l} = \frac{k!}{l!\,(k-l)!}$

Suppose we have figured out formulas for $S_0, S_1, S_2, \ldots, S_{k-1}$. We know (from the binomial theorem) that for any j,

$$(j+1)^{k+1} - j^{k+1} = \binom{k+1}{1} j^k + \binom{k+1}{2} j^{k-1} + \binom{k+1}{3} j^{k-2} + \ldots + 1$$

Write this out for each value $j = 1, 2, \ldots n$.

$$\begin{aligned}
2^{k+1} - 1^{k+1} &= \binom{k+1}{1} 1^k + \binom{k+1}{2} 1^{k-1} + \binom{k+1}{3} 1^{k-2} \ldots + 1 \\
3^{k+1} - 2^{k+1} &= \binom{k+1}{1} 2^k + \binom{k+1}{2} 2^{k-1} + \binom{k+1}{3} 2^{k-2} \ldots + 1 \\
&\ldots\ldots \\
(n+1)^{k+1} - n^{k+1} &= \binom{k+1}{1} n^k + \binom{k+1}{2} n^{k-1} + \binom{k+1}{3} n^{k-2} \ldots + 1.
\end{aligned}$$

<u>Add the columns:</u>

$$\begin{aligned}
(n+1)^{k+1} - 1 &= \binom{k+1}{1}(1^k + 2^k + \ldots + n^k) + \binom{k+1}{2}(1^{k-1} + 2^{k-1} + \ldots + n^{k-1}) \\
&\quad + \binom{k+1}{3}(1^{k-2} + 2^{k-2} + \ldots + n^{k-1}) \quad \ldots + n \\
&= \binom{k+1}{1} S_k + \binom{k+1}{2} S_{k-1} + \binom{k+1}{3} S_{k-2} + \ \ldots \ + S_0.
\end{aligned}$$

Then we solve for what we want:

$$\begin{aligned}
S_k &= [(n+1)^{k+1} - 1 - \binom{k+1}{2} S_{k-1} - \binom{k+1}{3} S_{k-2} - \ \ldots \ - S_0] \,/\, \binom{k+1}{1} \\
&= [(n+1)^{k+1} - 1 - \binom{k+1}{2} S_{k-1} - \binom{k+1}{3} S_{k-2} - \ \ldots \ - S_0] \,/\, (k+1)
\end{aligned}$$

We are assuming that we already have formulas for $S_{k-1}, S_{k-2}, \ldots S_1, S_0$ $-$ which we then substitute into this formula to get one closed, if complicated, formula for S_k in terms of n. Try it to find a formula for

$$S_4 = 1^4 + 2^4 + \ldots + n^4 = \ldots$$