Lecture 10

<u>Example</u>

Q1: Suppose $\lim_{x \to 3} \frac{x^2 - x - c}{x - 3} = 5$. What is the value of c?

A) $c = 0$

B) $c = -1$

C) $c = 4$

D) $c = 5$

E) $\boldsymbol{c = 6}$

Answer: To review, for $\lim_{x \to a} \frac{f(x)}{g(x)}$:

a) if $\lim_{x \to a} g(x) \neq 0$ (in the denominator), then $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$

b) if $\lim_{x \to a} g(x) = 0$ (in the denominator) and $\lim_{x \to a} f(x) =$ some number $k \neq \mathbf{0}$ (in the numerator), then $\lim_{x \to a} \frac{f(x)}{g(x)}$ does not exist (the fraction $\to \infty$ or $\to -\infty$, depending on the sign of thedenominator

Since $\lim_{x \to a} g(x) = 0$ in the question, then limit $\lim_{x \to a} \frac{f(x)}{g(x)} = 5$ could happen ONLY IF

$$\lim_{x \to 3} x^2 - x - c = 0, \text{ that is}$$

$$6 - c = 0, \text{ or } c = 6.$$

Notice that when $c = 6$, $\lim_{x \to 3} \frac{x^2 - x - 6}{x - 3} = \lim_{x \to 3} \frac{(x-3)(x+2)}{x - 3} = \lim_{x \to 3} (x + 2) = 5$, as desired.

From preceding lecture: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ (provided this limit exists)

If the limit exists, we say that f is differentiable at x. If the limits does not exist then we say that f has no derivative at x : that iws, f is not differentiable at x

Example

Q2: $\lim_{h \to 0} \frac{\sqrt{16+h} - 4}{h}$ is a derivative: that is, $\lim_{h \to 0} \frac{\sqrt{16+h} - 4}{h} = f'(a)$

for some function f and some value of a. What are f and a?

A) $f(x) = x + 16,\ a = 4$

B) $f(x) = \sqrt{x + 16},\ a = 4$

C) $f(x) = \sqrt{x},\ a = 4$

D) $\boldsymbol{f(x) = \sqrt{x},\ a = 16}$

E) $f(x) = \sqrt{x + 4},\ a = 4$

Answer:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \underset{\substack{\text{we want} \\ \downarrow}}{} \lim_{h \to 0} \frac{\sqrt{16+h} - 4}{h}$$

There is no specific "method" involved here, just trying to "invent" an f and an a that work. Comparing the two formulas we want 4 to match with $f(a)$, and the "$\sqrt{16 + h}$" piece suggests that the function f involves a $\sqrt{\quad}$.

If we try $f(x) = \sqrt{x}$ and $a = 16$, it works:

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\sqrt{16+h} - 4}{h}$$

So $\lim_{h \to 0} \frac{\sqrt{16+h} - 4}{h} = f'(16)$, where $f(x) = \sqrt{x}$

Example Find $f'(0)$, if possible, where $f(x) = |x|$

$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|h|}{h}$.

Break the limit into two parts to remove the absolute value sign:

$$\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{x \to 0^+} 1 = 1$$

$$\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} -1 = -1$$

Since $\lim_{h \to 0^+} \frac{|h|}{h} \neq \lim_{h \to 0^-} \frac{|h|}{h}$, we conclude that $\lim_{h \to 0} \frac{|h|}{h}$ does not exist.

That is, the function
$f(x) = \sqrt{x}$ has no derivative at 0 (it is not differentiable at 0)

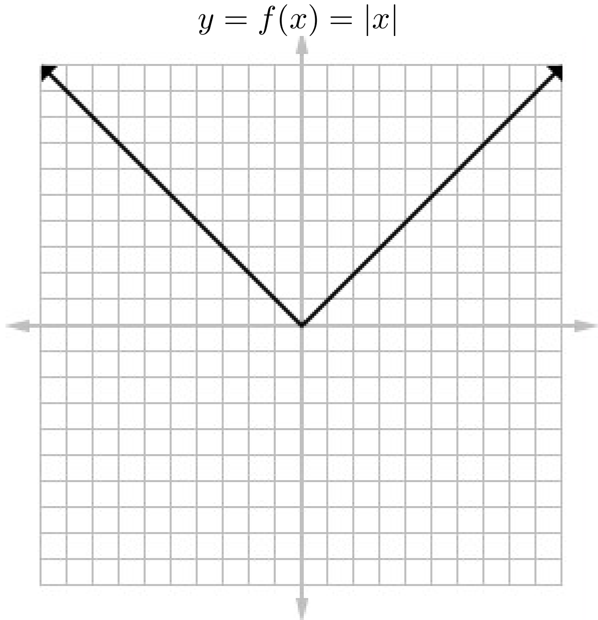
Intuitively, the reason is that the graph of $y = f(x) = |x|$ has a sharp corner when $x = 0$.

When $h < 0$, the secant line joining $(0,0)$ to a point $(h, |h|)$ on the "left half" of the graph always has slope -1, whereas

when $h < 0$, the secant line joining $(0,0)$ to a point $(h, |h|)$ on the "right half" of the graph always has slope 1.

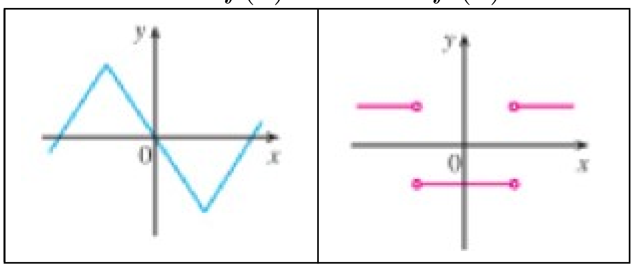
As $h \to 0$ the secant lines to not approach a tangent line:

$y = f(x) = |x|$



Example:

$f(x)$ $f'(x)$



Where there are sharp corners on the graph of $f(x)$, the function has no derivative. The derivative graph is not defined (there are open "o"s on the derivative graph where the jumps happen, but those didn't sow up clearly in the screen capture)

To the left of the first shapr corner, the graph of $f(x)$ has a constant positive slope (1, perhaps?) and so the graph of the derivative is constant (1 ?) for those x values.

A theorem

If $f(x)$ has a derivative at a **THEN** $\longrightarrow$ $f(x)$ must be continuous at a

That is, being differentiable at a is "better" or "stronger" statement about f than that f is continuous: differentiability at a forces the function automatically to also be continuous at a.

Why: Assume f does have a derivative at a. This means that $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a)$: this limit exists!

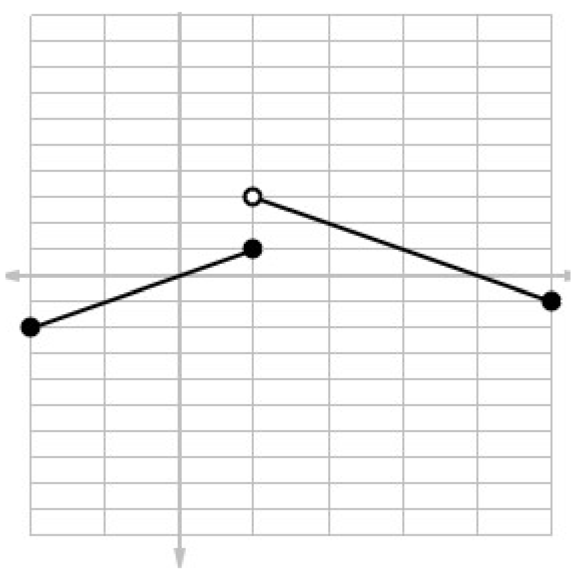
we know this limit exists

$$\text{So } \lim_{x\to a}(f(x)-f(a)) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot(x-a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot\lim_{x\to a}(x-a)$$

$$= f'(a)\cdot\lim_{x\to a}(x-a) = f'(a)\cdot 0 = 0$$

So $\lim_{x\to a}(f(x)-f(a)) = 0$, which is the same as saying that $\lim_{x\to a} f(x) = f(a)$. This means that f is continuous at a!

Example: The function pictured is not differentiable at 1 because it is not continuous as 1. (*The theorem says that if it were differentiable at 1, it would also have to be conitnuous at 1*).



IMPORTANT: Notice that the converse is NOT true.

If a function is continuous at a point, it might not be differentiable there.

$f(x)$ must have a derivative at a $\longleftarrow$ **NO!** $f(x)$ is continuous at a

For example (see picture above), $f(x) = |x|$ is continous at 0, but it is not differentiable at 0.

Example

Q3: Let $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$. Then $f'(0)$

A) $= \quad 0$

B) $= \quad 1$

C) $= -1$

D) $= \quad \frac{1}{3}$

E) DNE

Answer $\qquad f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h^{\frac{1}{3}} - 0}{h} = \lim_{h \to 0} \frac{1}{h^{2/3}}$

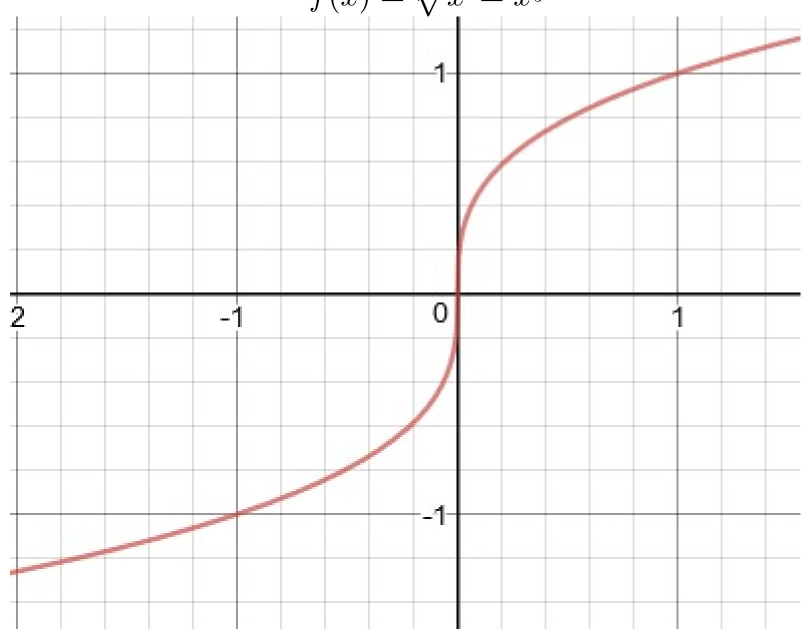
As $h \to 0$, $h^{\frac{2}{3}} = \sqrt[3]{h^2} \to 0$. Therefore $\lim_{h \to 0} \frac{1}{h^{2/3}}$ does not exist ($= \infty$)

(why ∞, not $-\infty$?)

So $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$ is not differentiable 0: the derivate at 0 does not exist.

Here (geometrically) the issue is not a "sharp corner:" as $h \to 0$ the secant lines joining $(0, 0)$ to another $(h.h^{\frac{1}{3}})$ approach a vertical tangent line, as a vertical line has no slope:

$f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$



Notice again: this function is continuous at 0 but not differentiable at 0.

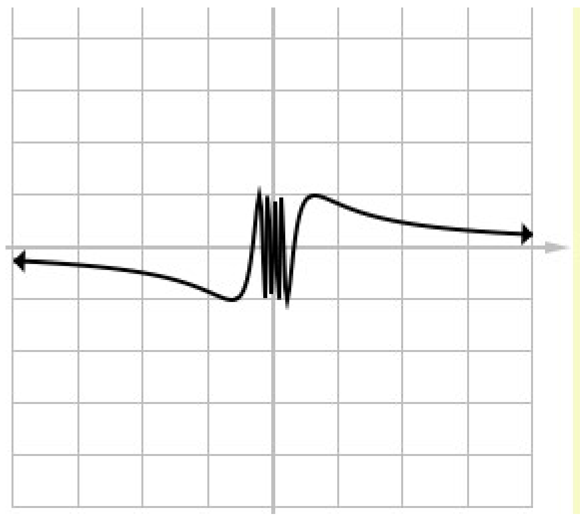
Example

Let $f(x) = \begin{cases} x\sin(\frac{1}{x}) & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}$

Is this function differentiable at 0? try to find the derivatice at 0:

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h\sin(\frac{1}{h})}{h} = \lim_{h \to 0} \sin(\tfrac{1}{h})$$

But $\lim_{h \to 0} \sin(\frac{1}{h})$ does not exist: the function simply oscillates up and down between ± 1 near 0: the following is the graph of $\sin(\frac{1}{h})$ as $h \to 0$:



The original functio $f(x)$ is not differentiable at 0 because of rapid oscillations. The graph of $f(x)$ looks like