

ORTHOGONAL MULTIPLICATIONS OF TYPE $[3, 4, p], p \leq 12$

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ABSTRACT. We describe the moduli space of orthogonal multiplications of type $[3, 4, p], p \leq 12$, and its application to the hypersurface theory.

1. INTRODUCTION

An orthogonal multiplication of type $[m, n, p], m \leq n$, is a bilinear map $F : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^p$ such that $|F(x, y)| = |x||y|$ for all x and y . The orthogonal multiplication is *full* if the image of F spans \mathbb{R}^p ; it follows that necessarily $n \leq p \leq mn$ when F is full. Alternatively, we may define $x \circ y := F(x, y)$, so that with respect to \circ , the orthogonal multiplication F may be thought of as a sort of norm-preserving product among the respective Euclidean spaces. The most well known of such “products” may be the ones of the normed algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and the octonion algebra \mathbb{O} . Conversely, Hurwitz [15] proved that, up to domain and range equivalence, an orthogonal multiplication of type $[l, l, l]$ satisfies $l = 1, 2, 4$, or 8 , and arises from the respective normed algebra product. This was generalized to the type $[l, m, m]$ by Radon [18], who showed that such an orthogonal multiplication exists if and only if $l \leq \rho(m)$, where $\rho(m)$ define the Hurwitz-Radon function. Adem [1], [2] classified orthogonal multiplications of type $[l, m, m + 1]$, while Gauchman and Toth [11], [12] classified the type $[l, m, m + 2]$.

In addition to the aforementioned, there have been many extensive studies of the admissibility problem of orthogonal multiplications, i.e., of the existence of a given type $[m, n, p]$, in the literature (see [19] and the comprehensive references therein). The case when $m = n$ relates in a particularly interesting way to geometry in that the Hopf map $\Phi : (x, y) \mapsto (2x \circ y, |x|^2 - |y|^2)$ from S^{2m-1} to S^p turns out to be harmonic, and this has spurred many investigations [9], [10], [14], [17], [20], [22], [23], [24], [25] (and the references therein), to mention just a few.

Along a different direction, there appears to have been only sporadic studies of the moduli space, up to domain and range equivalence, of the types $[m, n, p]$ when one fixes $m \leq n$ and varies p (it is understood that the orthogonal multiplications are full in each \mathbb{R}^p); note that necessarily $n \leq p \leq mn$. In this regard, we mention the work of Parker [16] when

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$m = n = 3$, Toth [21] for an elegant, intrinsic structural framework in the case $m = n$, Guo [13] in the case $m = 2$, and Toth [22] in the comprehensive case $m = n = 4$.

Lying between Parker's purely algebraic classification of $[3, 3, p]$, $p \leq 9$, of dimension 3, and Toth's moduli $[4, 4, p]$, $p \leq 16$, of dimension 24, is the moduli $[3, 4, p]$, $p \leq 12$, which the title of the present paper addresses. Its subspace consisting of the type $[3, 4, 8]$ plays a vital role in the classification, initiated by Cartan [3], [4] of isoparametric hypersurfaces with four principal curvatures in the sphere [8].

Slightly extending [21], we remark in Section 2 that the moduli space of orthogonal multiplications of type $[m, n, p]$, where $m \leq n$ are fixed and $n \leq p \leq mn$, up to range equivalence, can be identified with a compact convex set \mathcal{C} in $\wedge^2 \mathbb{R}^m \otimes \wedge^2 \mathbb{R}^n$ that is left fixed by the $SO(m) \otimes SO(n)$ -actions. Therefore, up to domain and range equivalence, the moduli space is $\mathcal{C}/SO(m) \otimes SO(n)$.

In the case of moduli space of type $[3, 4, p]$, $4 \leq p \leq 12$, to be denoted by $\mathcal{M}_{(3,4,12)}$ henceforth, it is more convenient to represent the 18-dimensional $\wedge^2 \mathbb{R}^3 \otimes \wedge^2 \mathbb{R}^4$ via the isomorphism

$$\wedge^2 \mathbb{R}^3 \otimes \wedge^2 \mathbb{R}^4 \simeq (\wedge^2 \mathbb{R}^3 \otimes \wedge^2 \mathbb{R}^3) \oplus (\wedge^2 \mathbb{R}^3 \otimes \wedge^2 \mathbb{R}^3)$$

by the fact that over \mathbb{R}^4 a 2-form is the direct sum of a self-dual and an anti-self-dual form, which can then be identified with a 3×6 matrix with two 3×3 blocks that we refer to as the Parker matrix $C = \begin{pmatrix} A & B \end{pmatrix}$.

For an orthogonal multiplication of type $[3, 4, p]$, $4 \leq p \leq 12$, its associated Parker matrix can be so normalized, through domain equivalence, that A is diagonal and B is upper triangular, leaving us with 9 independent variables that give rise to a 9-dimensional coarse fundamental domain of $\mathcal{M}_{(3,4,12)}$ (see (15) and (16)). With this, in Section 4, we show that the moduli of type $[3, 4, 8]$ has a rich structure carrying itself a grand moduli of dimension 5, denoted by $\mathcal{G}_{(3,4,8)}$, and an anomalous moduli $\mathcal{X}_{(3,4,8)}$ of dimension 3 that corresponds to the case when A and B in the normalized Parker matrix are both diagonal.

As is the case for any coarse fundamental domain, there is a boundary identification to "glue" the domain into the actual moduli space. We point out in Section 4 that these points are the ones, representing certain points in $\mathcal{G}_{(3,4,8)}$, for which at least one of the three off-diagonal entries of B of the Parker matrix is zero, where each of them is identified with one or two other points by a subgroup of S_3 , the permutation group on three letters, which in essence permutes the three off-diagonal entries of B without destroying the diagonal-upper-triangular pattern of A and B . As an application of this observation, in Section 5, we consider the situation when the range equivalence is more rigidly restricted to a fixed decomposition $\mathbb{R}^8 = \mathbb{R}^4 \oplus \mathbb{R}^4$, so that only $SO(4) \oplus SO(4) \subset SO(8)$ is allowed. This is pertinent to the classification of isoparametric hypersurfaces with four principal curvatures and multiplicity pair $(7, 8)$ in S^{31} , to which a certain orthogonal multiplication

of type $[3, 4, 8]$ is associated that respects the decomposition of \mathbb{R}^8 into two copies of \mathbb{R}^4 intrinsic to the isoparametric structure [8, Section 7]. We prove in Section 5 that all such orthogonal multiplications of type $[3, 4, 8]$ come from the grand moduli $\mathcal{G}_{(3,4,8)}$, a property, indicated in an *ad hoc* fashion in [8], crucial for the classification of the isoparametric hypersurfaces in S^{31} (see the ending paragraph of Section 5).

We conclude in Section 6 that $\mathcal{M}_{(3,4,12)}$ consists of the generic open set of dimension 9, and for each $0 \leq n \leq 4$, the generic moduli of type $[3, 4, 8 + n]$ is of dimension $5 + n$, which is achieved by studying the perturbation, via the normal exponential map, of the anomalous moduli $\mathcal{X}_{(3,4,8)}$. Inside $\mathcal{X}_{(3,4,8)}$ there sits a 1-dimensional moduli of type $[3, 4, 7]$, which degenerates to the quaternion multiplication (of type $[3, 4, 4]$). In particular, there are no orthogonal multiplications of type $[3, 4, 5]$ and $[3, 4, 6]$; we remark that this is also a consequence of the aforementioned general results in [1], [2], [11], [12].

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2. THE MODULI SPACE OF ORTHOGONAL MULTIPLICATIONS

In [21], an elegant, intrinsic geometric picture is given to capture the moduli space of all orthogonal multiplications of type $[m, m, p]$ for fixed m and for varying p , $m \leq p \leq m^2$. In fact, this is completely general for the moduli space of orthogonal multiplications of type $[m, n, p]$, where $m \leq n$ are fixed and $n \leq p \leq mn$, which we briefly sketch.

A full orthogonal multiplication $F : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^p$, $m \leq n$, satisfies $n \leq p \leq mn$ for dimension reasons. Accordingly, we assume $p \leq mn$.

Let e_1, \dots, e_m be an orthonormal basis for \mathbb{R}^m and let f_1, \dots, f_n be an orthonormal basis for \mathbb{R}^n . We identify \mathbb{R}^{mn} with $\mathbb{R}^m \otimes \mathbb{R}^n$. We set

$$x \circ y = F(x, y).$$

Following [16], we define

$$F_{ij,kl} = \langle e_i \circ f_j, e_k \circ f_l \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product. We have the properties

$$\begin{aligned} F_{ij,kl} &= F_{kl,ij}, \quad \forall i, j, k, l, \\ F_{ij,il} &= F_{li,ji} = 0, \quad j \neq l, \\ F_{ij,kl} &= -F_{il,kj}, \quad i \neq k, \\ F_{ij,kl} &= -F_{kj,il}, \quad j \neq l, \end{aligned} \tag{1}$$

from which the quantity

$$[ij, kl] := F_{ik,jl} \tag{2}$$

satisfies

$$[ij, kl] = -[ji, kl], \quad [ij, kl] = -[ij, lk].$$

Intrinsically, this means that if we define

$$\begin{aligned} g &:= \sum_{(ij,kl)} F_{ij,kl} (e_i \otimes f_j) \odot (e_k \otimes f_l), \\ \iota &:= \sum_{(ij,ij)} (e_i \otimes f_j) \odot (e_i \otimes f_j), \end{aligned} \tag{3}$$

in $S^2(\mathbb{R}^m \otimes \mathbb{R}^n)$, set

$$W := \text{span}((a \otimes b) \odot (a \otimes b)) \subset S^2(\mathbb{R}^m \otimes \mathbb{R}^n),$$

and let W^\perp be the space perpendicular to W in $S^2(\mathbb{R}^m \otimes \mathbb{R}^n)$, then by (1),

$$g - \iota \in W^\perp.$$

Moreover, if we let

$$g - \iota := c = \sum_{(ij,kl)} c_{ij,kl} (e_i \otimes f_j) \odot (e_k \otimes f_l), \tag{4}$$

then $c_{ij,kl}$ satisfies

$$c_{ij,kl} = [ik, jl].$$

Conversely, for any element c in W^\perp , the symmetric tensor

$$g := \iota + c := \sum_{(ij,kl)} F_{ij,kl} (e_i \otimes f_j) \odot (e_k \otimes f_l)$$

has the same properties as in (1) so that

$$[ik, jl] := F_{ij,kl}$$

satisfies

$$[ij, kl] = -[ji, kl], \quad [ij, kl] = -[ij, lk].$$

As a consequence,

$$W^\perp \simeq \wedge^2 \mathbb{R}^m \otimes \wedge^2 \mathbb{R}^n.$$

Meanwhile, each $c \in W^\perp$ induces a traceless symmetric endomorphism

$$c^* : \mathbb{R}^m \otimes \mathbb{R}^n \rightarrow \mathbb{R}^m \otimes \mathbb{R}^n \quad c^* : e_i \otimes f_j \mapsto \sum_{kl} c_{ij,kl} e_k \otimes f_l,$$

In particular, when c is induced by an orthogonal multiplication F as in (3), we have the extra property that for

$$X = \sum_{ij} X_{ij} e_i \otimes f_j \in \mathbb{R}^m \otimes \mathbb{R}^n,$$

there follows

$$\langle (id + c^*)(X), X \rangle = \sum_{ij,kl} F_{ij,kl} X_{ij} X_{kl} = \left\langle \sum_{ij} X_{ij} e_i \circ f_j, \sum_{ij} X_{ij} e_i \circ f_j \right\rangle \geq 0,$$

so that $\iota + c$ regarded as a symmetric endomorphism is positive semi-definite. To recover $x \circ y$, by the universal property of tensor product, there is a unique

endomorphism $D : \mathbb{R}^m \otimes \mathbb{R}^n \rightarrow \mathbb{R}^p \subset \mathbb{R}^m \otimes \mathbb{R}^n$ such that $x \circ y = D(x \otimes y)$. We have

$$F_{ij,kl} = \langle e_i \circ f_j, e_k \circ f_l \rangle = \langle D^{tr} D(e_i \otimes f_j), e_k \otimes f_l \rangle.$$

That is, by (3),

$$D^{tr} D = g^*,$$

where g^* is the associated endomorphism of g .

Conversely, given a positive semi-definite symmetric $g = \iota + c$, let $g^* = id + c^*$ be the associated endomorphism. Let E_0 be the 0-eigen space of g^* , and let E_0^\perp be its orthogonal complement in $\mathbb{R}^m \otimes \mathbb{R}^n$. Let n_1, \dots, n_p be eigenvectors of E_0^\perp with eigenvalues $\lambda_1, \dots, \lambda_p > 0$. For any p orthonormal elements v_1, \dots, v_p in $\mathbb{R}^m \otimes \mathbb{R}^n$ define

$$D_v : \mathbb{R}^m \otimes \mathbb{R}^n \rightarrow \mathbb{R}^m \otimes \mathbb{R}^n, \quad n_i \mapsto \sqrt{\lambda_i} v_i, \quad E_0 \mapsto 0.$$

Then D_v satisfies

$$D_v^{tr} D_v = g^*, \tag{5}$$

and, moreover, this exhausts all possible D_v satisfying $D_v^{tr} D_v = g^*$. We define

$$e_i \circ_v f_j := D_v(e_i \otimes f_j) \in \text{span}(v_1, \dots, v_p) \simeq \mathbb{R}^p,$$

from which it is checked by (5) that for $|x| = |y| = 1$

$$|x \circ_v y|^2 = \sum_{(ij,kl)} x_i y_j x_k y_l \langle D_v(e_i \otimes f_j), D_v(e_k \otimes f_l) \rangle = \sum_{(ij,kl)} F_{ij,kl} x_i y_j x_k y_l = 1,$$

where we set $x = e_1$ and $y = s e_1 + t e_2, s^2 + t^2 = 1$, for a convenient calculation by (1). In other words, $F_v(x, y) = x \circ_v y$ so defined is an orthogonal multiplication. Note that D_v differs from D_w only by an orthonormal basis change in $\mathbb{R}^m \otimes \mathbb{R}^n$, so that they induce the same orthogonal multiplication up to range equivalence.

As a result, up to range equivalence, the set of orthogonal multiplications is a compact convex subset \mathcal{C} (call it the *preliminary moduli space* in the following) of $W^\perp \simeq \wedge^2 \mathbb{R}^m \otimes \wedge^2 \mathbb{R}^n$, consisting of c for which $\iota + c$ is positive semi-definite. Its interior consists of full orthogonal multiplications, and the non-full orthogonal multiplications lie on the boundary.

Since the domain equivalence $SO(m) \otimes SO(n)$ acts on W^\perp and fixes \mathcal{C} , the moduli space of orthogonal multiplications of type $[m, n, p]$, up to domain and range equivalence, where $m \leq n$ are fixed and $n \leq p \leq mn$, is $\mathcal{C}/SO(m) \otimes SO(n)$.

3. TYPE $[2, 4, 8]$

We recall the result in [13], specialized to our situation with a slightly different approach, to gain motivation for the subsequent development. Since the preliminary moduli space $\mathcal{C} \subset \wedge^2 \mathbb{R}^2 \otimes \wedge^2 \mathbb{R}^4 = \wedge^2 \mathbb{R}^4$, this case simply says if we choose the basis elements u_5, \dots, u_8 to be $e_2 \circ f_1, \dots, e_2 \circ f_4$, and

complete the basis to $u_1, \dots, u_4, u_5, \dots, u_8$, then the two Hurwitz matrices defined by

$$F_a = \text{the matrix whose } b\text{th row is } e_a \circ f_b, \quad 1 \leq a \leq 2,$$

are

$$F_1 = (c_1 \quad w_1), \quad F_2 = (0 \quad Id),$$

where the 4×4 w_1 is skew-symmetric and gives the above $\wedge^2 \mathbb{R}^4$. Knowing that the connected \mathcal{C} is six-dimensional, we can immediately find its structure:

$$F_1 = (UDU^{tr} \quad UKU^{tr}), \quad D := \begin{pmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \sigma_2 & 0 \\ 0 & 0 & 0 & \sigma_1 \end{pmatrix}, \quad K := \begin{pmatrix} 0 & 0 & 0 & -\mu \\ 0 & 0 & -\nu & 0 \\ 0 & \nu & 0 & 0 \\ \mu & 0 & 0 & 0 \end{pmatrix},$$

$$U \in SO(4), \quad \sigma_1^2 + \mu^2 = 1, \quad \sigma_2^2 + \nu^2 = 1.$$

where the adjoint orbit of K by U is only 4-dimensional.

It is easily checked that F_1 so given and $F_2 := (0 \quad I)$ form a Hurwitz system of dimension 6. Conversely, every F_1 can be brought to this form by appropriate coordinate changes. This is because the skew-symmetry of w_1 allows us to find an orthogonal matrix U such that $w_1 = UKU^{tr}$, so that by performing a coordinate change on $\text{span}(f_1, \dots, f_4)$ and on $\text{span}(u_5, \dots, u_8)$, leaving F_2 fixed, we may assume $F_1 = (U^{tr} c_1 \quad K)$. Since

$$(U^{tr} c_1)(U^{tr} c_1)^{tr} + KK^{tr} = Id,$$

it follows that the first term in the sum is diagonal, or rather, the rows of $U^{tr} c_1$ are mutually orthogonal, so that we may find an orthogonal matrix W such that $U^{tr} c_1 = DW$. Hence, after a coordinate change on $\text{span}(u_1, \dots, u_4)$, fixing F_2 , we may assume

$$F_1 = (D \quad K), \quad F_2 = (0 \quad Id). \quad (6)$$

These constitute the moduli space of type $[2, 4, p], p \leq 8$, through the action $SO(2) \otimes SO(4)$ on $\wedge^2 \mathbb{R}^2 \otimes \wedge^2 \mathbb{R}^4$, where the left action is trivial while the right one is the adjoint action.

4. TYPE $[3, 4, 8]$

4.1. The setup. Given an orthogonal multiplication F of type $[3, 4, 8]$, its Hurwitz matrices $F_a, 1 \leq a \leq 3$, defined by

$$F_a := \text{the matrix whose } b\text{th row is } e_a \circ f_b$$

with $x \circ y := F(x, y)$, satisfies the Hurwitz condition

$$F_a F_b^{tr} + F_b F_a^{tr} = 2\delta_{ab} Id, \quad 1 \leq a, b \leq 3. \quad (7)$$

We can choose the ‘‘anchor’’ matrix to be

$$F_3 = (0 \quad Id)$$

as in type $[2, 4, 8]$. We seek to express F_1 and F_2 in the most convenient form for analysis. (6) will be our guide to formulate F_1 . To achieve the goal, recall (2) and consider the 3×6 matrix

$$C := ([ij, kl]), \quad (8)$$

referred to henceforth as the Parker matrix [16], which defines a linear map

$$C : \wedge^2(\mathbb{R}^4) \rightarrow \wedge^2(\mathbb{R}^3). \quad (9)$$

We identify the basis of the target space by

$$e_1 \wedge e_2, \quad e_3 \wedge e_1, \quad e_3 \wedge e_2,$$

and identify the domain space by

$$\wedge^2(\mathbb{R}^4) = \wedge^2(\mathbb{R}^3) \oplus \wedge^2(\mathbb{R}^3)$$

where the right hand side consists of the space of self-dual forms and of anti-self-dual forms, where the former is spanned by

$$f_1 \wedge f_2 + f_3 \wedge f_4, \quad f_1 \wedge f_3 + f_4 \wedge f_2, \quad f_1 \wedge f_4 + f_2 \wedge f_3,$$

and the latter by

$$f_1 \wedge f_2 - f_3 \wedge f_4, \quad f_1 \wedge f_3 - f_4 \wedge f_2, \quad f_1 \wedge f_4 - f_2 \wedge f_3.$$

As a consequence

$$C = \begin{pmatrix} A & B \end{pmatrix}, \quad (10)$$

where A and B are of size 3×3 .

As in [16], A and B can be written in the form

$$A = U^{tr} D_1 W_1 U, \quad B = V^{tr} D_2 W_2 V,$$

where U, V, W_1, W_2 are orthogonal and D_1, D_2 are diagonal, so that by applying orthogonal changes to row and column spaces, we may assume

$$C = \begin{pmatrix} D & T \end{pmatrix} \quad (11)$$

with D diagonal and T upper triangular. This means that we may assume

$$\begin{aligned} [31, 12 \pm 34] &= [32, 12 \pm 34] = [32, 13 \pm 42] = 0, \\ [12, 13 + 42] &= [12, 14 + 23] = [31, 14 + 23] = 0; \end{aligned} \quad (12)$$

in other words, we now have the identities

$$\begin{aligned} F_{31,12} = F_{33,14} = 0, \quad F_{31,22} = F_{33,24} = 0, \quad F_{31,23} = F_{34,22} = 0, \\ F_{11,23} + F_{14,22} = F_{11,24} + F_{12,23} = F_{31,14} + F_{32,13} = 0. \end{aligned} \quad (13)$$

With this choice of basis, we can now pick the orthonormal set

$$u_5 := e_3 \circ f_1, \quad u_6 := e_3 \circ f_2, \quad u_7 := e_3 \circ f_3, \quad u_8 := e_3 \circ f_4,$$

relative to which we have

$$F_3 = \begin{pmatrix} c_3 & w_3 \end{pmatrix} = \begin{pmatrix} 0 & Id \end{pmatrix}. \quad (14)$$

It is then calculated that the orthogonal projection of

$$v_1 := e_2 \circ f_1, \quad v_2 := e_2 \circ f_2, \quad v_3 := e_2 \circ f_3, \quad v_4 := e_2 \circ f_4$$

onto $\text{span}(u_5, \dots, u_8)$, denoted $v_1^\perp, \dots, v_4^\perp$, are, respectively,

$$v_1^\perp = F_{21,34} u_8, \quad v_2^\perp = F_{22,33} u_7, \quad v_3^\perp = F_{23,32} u_6, \quad v_4^\perp = F_{24,31} u_5,$$

which are mutually orthogonal with

$$|v_1| = |v_4|, \quad |v_2| = |v_3|,$$

so that we may complete the orthonormal basis u_1, \dots, u_8 by setting

$$u_i := (e_2 \circ f_i - v_i^\perp) / (\text{length}), \quad 1 \leq i \leq 4.$$

Consequently, relative to u_1, \dots, u_8 we have

$$F_1 = (c_1 \quad w_1), \quad c_1 = \begin{pmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \sigma_2 & 0 \\ 0 & 0 & 0 & \sigma_1 \end{pmatrix}, \quad w_1 = \begin{pmatrix} 0 & 0 & 0 & -\mu \\ 0 & 0 & -\nu & 0 \\ 0 & \nu & 0 & 0 \\ \mu & 0 & 0 & 0 \end{pmatrix},$$

$$\mu := F_{24,31}, \quad \nu := F_{23,32}, \quad \sigma_1 := \sqrt{1 - \mu^2}, \quad \sigma_2 := \sqrt{1 - \nu^2}. \quad (15)$$

Now, $e_1 \circ f_1, \dots, e_1 \circ f_4$ can be expanded in terms of u_1, \dots, u_8 to yield $F_2 = (c_2 \quad w_2)$, where

$$c_2 = \begin{pmatrix} -\beta\mu/\sigma_1 & (F_{11,22} - \alpha\nu)/\sigma_2 & F_{11,23}/\sigma_2 & F_{11,24}/\sigma_1 \\ (F_{1221} - \mu\gamma)/\sigma_1 & \beta\nu/\sigma_2 & F_{12,23}/\sigma_2 & F_{12,24}/\sigma_1 \\ F_{13,21}/\sigma_1 & F_{13,22}/\sigma_2 & \beta\nu/\sigma_2 & (F_{13,24} - \alpha\mu)/\sigma_1 \\ F_{14,21}/\sigma_1 & F_{14,22}/\sigma_2 & (F_{14,23} - \gamma\nu)/\sigma_2 & -\beta\mu/\sigma_1 \end{pmatrix},$$

$$w_2 = \begin{pmatrix} 0 & 0 & -\alpha & -\beta \\ 0 & 0 & \beta & -\gamma \\ \alpha & -\beta & 0 & 0 \\ \beta & \gamma & 0 & 0 \end{pmatrix}, \quad \alpha := F_{31,13}, \quad \beta = F_{31,14}, \quad \gamma := F_{32,14}. \quad (16)$$

Note that when either $\sigma_1 = 0$ or $\sigma_2 = 0$, the corresponding columns for c_2 give, respectively,

$$\begin{aligned} \beta = 0, \quad F_{11,22} = -\gamma\mu, \quad F_{14,23} = -\alpha\mu, \quad F_{14,21} = F_{14,22} = 0, \quad \text{if } \sigma_1 = 0, \\ \beta = 0, \quad F_{11,22} = \alpha\nu, \quad F_{14,23} = \gamma\nu, \quad F_{14,21} = F_{14,22} = 0, \quad \text{if } \sigma_2 = 0. \end{aligned} \quad (17)$$

This belongs to the most degenerate case we will consider later. As a corollary, we have

Lemma 1. *If $\beta \neq 0$, then $|\mu|, |\nu| < 1$. Equivalently, $\sigma_1\sigma_2 \neq 0$.*

It is then legitimate to perform operations with the entries in c_2 when $\beta \neq 0$.

With the normalization of the Parker matrix given in (12), or equivalently, in (13), we have chosen a coarse fundamental domain for the moduli space of orthogonal multiplications of type [3, 4, 8] in (15) and (16), up to domain and range equivalence. There remains a boundary identification of the coarse fundamental domain to be addressed in Section 4.3.

Note that the Hurwitz condition (7) is now

$$c_a c_b^{tr} + c_b c_a^{tr} + w_a w_b^{tr} + w_b w_a^{tr} = 2\delta_{ab} Id, \quad 1 \leq a, b \leq 3.$$

4.2. The generic case when $\beta \neq 0$.

Lemma 2. *Assume $\beta \neq 0$. If $F_{14,22} \neq 0$, then*

$$\alpha = \gamma, \quad \mu = -\nu, \quad F_{11,22} = F_{14,23}.$$

Proof. Since the rows of F_2 are mutually orthogonal, of which the first and the fourth, and, respectively, the second and the third, give

$$F_{14,22} ((F_{11,22} - F_{14,23}) + \nu(\gamma - \alpha)) = 0, \quad F_{14,22} ((F_{11,22} - F_{14,23}) - \mu(\gamma - \alpha)) = 0, \quad (18)$$

from which there results

$$(\mu + \nu)(\gamma - \alpha) = 0,$$

so that either $\alpha = \gamma$ or $\mu = -\nu$.

If $\mu = -\nu$, then $\sigma_1 = \sigma_2 := \sigma$ and the first and the second rows of F_2 give

$$0 = \beta F_{11,22}(\mu + \nu) + \beta(\mu^2 + \sigma^2)(\gamma - \alpha) = \beta(\gamma - \alpha), \quad \mu^2 + \sigma^2 = 1, \quad (19)$$

we obtain $\alpha = \gamma$.

On the other hand, if $\alpha = \gamma$, then (18) gives $F_{11,22} = F_{14,23}$. Meanwhile, the rows of c_2 are now mutually orthogonal and of equal length, which implies that the columns of c_2 are mutually orthogonal and of the same length, of which the first and the second give

$$\beta F_{11,22}(\mu + \nu) = 0.$$

That is, $\mu = -\nu$ when $F_{11,22} = F_{14,23} \neq 0$.

Suppose now $F_{11,22} = F_{14,23} = 0$. Since the third and fourth rows of c_2 are orthogonal, we obtain

$$(F_{14,21}F_{14,22} + \alpha\beta) \frac{\mu^2 - \nu^2}{\sigma_1^2 \sigma_2^2} = 0,$$

so that

$$F_{14,21}F_{14,22} = -\alpha\beta \quad (20)$$

if $\mu^2 \neq \nu^2$. Similarly, the second and the fourth rows of c_2 give

$$\alpha F_{14,21} \left(\frac{\mu}{\sigma_1^2} + \frac{\nu}{\sigma_2^2} \right) = \beta F_{14,22} \left(\frac{\mu}{\sigma_1^2} + \frac{\nu}{\sigma_2^2} \right),$$

where

$$\frac{\mu}{\sigma_1^2} + \frac{\nu}{\sigma_2^2} = \frac{(u+v)(1-uv)}{\sigma_1^2 \sigma_2^2} \neq 0.$$

Therefore, we derive

$$\alpha F_{14,21} = \beta F_{14,22},$$

which, upon substituting (20), results in

$$(\beta F_{14,22})^2 = \alpha\beta F_{14,21}F_{14,22} = -(\alpha\beta)^2.$$

It follows that

$$\alpha\beta = 0, \quad \beta F_{14,22} = 0,$$

so that $F_{14,22} = 0$, contradicting $F_{14,22} \neq 0$, and thus $\mu^2 = \nu^2$. So, after a coordinate sign change, we may assume $\mu = -\nu$, from which (19) results in $\alpha = \gamma$. \square

Lemma 3. *Assume $\beta \neq 0$. If $F_{14,22} = 0$ and $F_{14,21} \neq 0$, then*

$$\alpha = \gamma, \quad \mu = -\nu, \quad F_{11,22} = F_{14,23}.$$

Proof. With $F_{14,22} = 0$, the first and the second rows, and respectively the third and the fourth rows, give, after cancelling β ,

$$\begin{aligned} F_{11,22} \left(\frac{\mu}{\sigma_1^2} + \frac{\nu}{\sigma_2^2} \right) + \left(\frac{\mu^2 \gamma}{\sigma_1^2} - \frac{\nu^2 \alpha}{\sigma_2^2} \right) + (\gamma - \alpha) &= 0, \\ F_{14,23} \left(\frac{\mu}{\sigma_1^2} + \frac{\nu}{\sigma_2^2} \right) + \left(\frac{\mu^2 \alpha}{\sigma_1^2} - \frac{\nu^2 \gamma}{\sigma_2^2} \right) + (\alpha - \gamma) &= 0. \end{aligned} \quad (21)$$

Meanwhile, the first and the third rows, and, respectively, the second and the fourth rows, give, after cancelling $F_{14,21}$,

$$\begin{aligned} \frac{F_{11,22}}{\sigma_2^2} - \frac{F_{14,23}}{\sigma_1^2} &= \alpha \left(\frac{\mu}{\sigma_1^2} + \frac{\nu}{\sigma_2^2} \right), \\ \frac{F_{11,22}}{\sigma_1^2} - \frac{F_{14,23}}{\sigma_2^2} &= -\gamma \left(\frac{\mu}{\sigma_1^2} + \frac{\nu}{\sigma_2^2} \right). \end{aligned} \quad (22)$$

We substitute (21) into (22) to come up with, respectively,

$$\alpha (2 - (\mu + \nu)^2) = 2\gamma, \quad \gamma (2 - (\mu + \nu)^2) = 2\alpha$$

using $\mu^2 + \sigma_1^2 = \nu^2 + \sigma_2^2 = 1$, from which we conclude that

$$\text{either } \alpha = \gamma = 0, \quad \text{or } 2 - (\mu + \nu)^2 = \pm 2.$$

However, since $|\mu| + |\nu| < 2$ by Lemma 1, it is impossible that $2 - (\mu + \nu)^2 = -2$. There follows

$$\text{either } \alpha = \gamma = 0, \quad \text{or } \mu + \nu = 0.$$

In the former case, c_2 is orthogonal up to a constant, so that its first and second columns are of some length l , which amounts to

$$\beta^2 \mu^2 + F_{11,22}^2 + F_{14,21}^2 = l^2 \sigma_1^2, \quad \beta^2 \nu^2 + F_{11,22}^2 + F_{14,21}^2 = l^2 \sigma_2^2,$$

so that

$$\beta^2 (\mu^2 - \nu^2) = l^2 (\sigma_1^2 - \sigma_2^2) = -l^2 (\mu^2 - \nu^2), \quad \text{or } (\beta^2 + l^2) (\mu^2 - \nu^2) = 0.$$

That is, with $\beta \neq 0$, we have

$$\mu^2 = \nu^2, \quad (23)$$

and we may assume $\mu = -\nu$ by a coordinate sign change. Now that $\mu = -\nu$ in either case, (21) then gives $\alpha = \gamma$, and (22) gives $F_{11,22} = F_{14,23}$ since now $\sigma_1 = \sigma_2$. \square

Lemma 4. *Assume $\beta \neq 0$. If $F_{14,21} = F_{14,22} = 0$, then*

$$\alpha = \gamma, \quad \mu = -\nu.$$

Moreover, either $F_{14,23} = F_{11,22}$, or $F_{14,23} = -F_{11,22} - 2\alpha\mu$.

Proof. First note that (21) continues to hold, so that we obtain

$$\begin{aligned} (F_{11,22} - \alpha\nu)\left(\frac{\mu}{\sigma_1^2} + \frac{\nu}{\sigma_2^2}\right) &= \frac{-\mu(\mu\gamma + \alpha\nu)}{\sigma_1^2} - (\gamma - \alpha), \\ (F_{11,22} + \mu\gamma)\left(\frac{\mu}{\sigma_1^2} + \frac{\nu}{\sigma_2^2}\right) &= \frac{\nu(\mu\gamma + \alpha\nu)}{\sigma_2^2} - (\gamma - \alpha), \\ (F_{14,23} + \alpha\mu)\left(\frac{\mu}{\sigma_1^2} + \frac{\nu}{\sigma_2^2}\right) &= \frac{\nu(\mu\alpha + \gamma\nu)}{\sigma_2^2} - (\alpha - \gamma), \\ (F_{14,23} - \gamma\nu)\left(\frac{\mu}{\sigma_1^2} + \frac{\nu}{\sigma_2^2}\right) &= \frac{-\mu(\mu\alpha + \gamma\nu)}{\sigma_1^2} - (\alpha - \gamma). \end{aligned} \tag{24}$$

Meanwhile, the four rows of F_2 being of length 1 translates to

$$\begin{aligned} \frac{(F_{11,22} - \alpha\nu)^2}{\sigma_2^2} &= 1 - \alpha^2 - \frac{\beta^2}{\sigma_1^2}, \\ \frac{(F_{11,22} + \gamma\mu)^2}{\sigma_1^2} &= 1 - \gamma^2 - \frac{\beta^2}{\sigma_2^2}, \\ \frac{(F_{14,23} + \alpha\mu)^2}{\sigma_1^2} &= 1 - \alpha^2 - \frac{\beta^2}{\sigma_2^2}, \\ \frac{(F_{14,23} - \gamma\nu)^2}{\sigma_2^2} &= 1 - \gamma^2 - \frac{\beta^2}{\sigma_1^2}, \end{aligned} \tag{25}$$

from which we derive

$$\frac{(F_{11,22} + \gamma\mu)^2}{\sigma_1^2} - \frac{(F_{14,23} + \alpha\mu)^2}{\sigma_1^2} = \frac{(F_{14,23} - \gamma\nu)^2}{\sigma_2^2} - \frac{(F_{11,22} - \alpha\nu)^2}{\sigma_2^2}. \tag{26}$$

Inserting (24) into (26) and cancelling $\sigma_1^2\sigma_2^2$ yields

$$\begin{aligned} &\frac{(\mu\alpha(\mu + \nu) + (\gamma - \alpha))^2}{\sigma_1^2} - \frac{(-\mu\gamma(\mu + \nu) + (\gamma - \alpha))^2}{\sigma_1^2} \\ &= \frac{(\alpha\nu(\mu + \nu) + (\gamma - \alpha))^2}{\sigma_2^2} - \frac{(\nu\gamma(\mu + \nu) - (\gamma - \alpha))^2}{\sigma_2^2}. \end{aligned}$$

This further simplifies to

$$(\mu + \nu)(\alpha^2 - \gamma^2)(\mu^2(\mu + \nu) - 2\mu)\sigma_2^2 = (\mu + \nu)(\alpha^2 - \gamma^2)(\nu^2(\mu + \nu) - 2\nu)\sigma_1^2,$$

so that it finally arrives at

$$(\mu^2 - \nu^2)(\alpha^2 - \gamma^2)(\mu^2 - \nu^2 - 2) = 0. \tag{27}$$

As a consequence, (27) gives, since $\mu^2 - \nu^2 - 2 \neq 0$ by Lemma 1, that

$$\text{either } \alpha^2 = \gamma^2, \quad \text{or, } \mu^2 = \nu^2.$$

If $\mu^2 = \nu^2$, then we may assume $\mu = -\nu$ by a coordinate sign change; with $\sigma = \sigma_1 = \sigma_2$ now, this implies by (21)

$$\left(\frac{\mu^2}{\sigma^2} + 1\right)(\gamma - \alpha) = 0, \quad (28)$$

so that $\alpha = \gamma$.

If $\alpha^2 = \gamma^2$, then c_2 is orthogonal up to a constant. Hence, the same argument leading to (23) results in $\mu^2 = \nu^2$. By a coordinate sign change, we may assume $\mu = -\nu$, so that once more (28) gives $\alpha = \gamma$.

Now that $\alpha = \gamma$ and $\mu = -\nu$, (21) becomes void. The only essential equations left are the first and the fourth ones in (25), which read

$$(F_{11,22} + \alpha\mu)^2 = (F_{14,23} + \alpha\mu)^2 = (1 - \mu^2)(1 - \alpha^2) - \beta^2.$$

Therefore,

$$\text{either } F_{11,22} + F_{14,23}, \text{ or } F_{14,23} = -F_{11,22} - 2\alpha\mu.$$

In both cases, β satisfies the constraint

$$\beta^2 = (1 - \mu^2)(1 - \alpha^2) - (F_{11,22} + \alpha\mu)^2.$$

□

We summarize the above analysis in the following proposition.

Proposition 1. *Assume $\beta \neq 0$. Then*

$$\alpha = \gamma, \quad \mu = -\nu. \quad (29)$$

Moreover, if either $F_{14,21}$ or $F_{14,22}$ is nonzero, then we have the further constraint

$$F_{11,22} = F_{14,23}. \quad (30)$$

The moduli of such orthogonal multiplications of type [3, 4, 8] depends on the five variables $\alpha, \mu, F_{14,21}, F_{14,22}, F_{11,22}$, while β is linked with the five variables by

$$\beta^2 = (1 - \mu^2) \left((1 - \alpha^2) - (F_{11,22} + \alpha\mu)^2 - F_{14,21}^2 - F_{14,22}^2 \right). \quad (31)$$

On the other hand, if $F_{14,21} = F_{14,22} = 0$, then

$$\text{either } F_{11,22} = F_{14,23}, \text{ or } F_{14,23} = -F_{11,22} - 2\alpha\mu. \quad (32)$$

Both cases satisfy (31) when we set $F_{14,21} = F_{14,22} = 0$.

Definition 1. *We will refer to the moduli represented by orthogonal multiplications of type [3, 4, 8], where $\alpha = \gamma$ and $\mu = -\nu$, as the **grand moduli** $\mathcal{G}_{(3,4,8)}$.*

Note that the condition $\alpha = -\beta$ and $\mu = \nu$ are equivalent to the one given in the definition up to a coordinate sign change.

In conclusion, all orthogonal multiplications with $\beta \neq 0$ are in the grand moduli $\mathcal{G}_{(3,4,8)}$, up to domain and range equivalence.

4.3. $\beta = 0$, the generic case when $F_{14,21}^2 + F_{14,22}^2 \neq 0$. The element of $SO(4)$ that fixes f_1 , maps f_2 to its negative, and interchanges f_3 and f_4 , so that it transforms the self-dual forms to self-dual forms as follows,

$$\begin{aligned} f_1 \wedge f_2 + f_3 \wedge f_4 &\mapsto -(f_1 \wedge f_2 + f_3 \wedge f_4), \\ f_1 \wedge f_3 + f_4 \wedge f_2 &\mapsto f_1 \wedge f_4 + f_2 \wedge f_3, \\ f_1 \wedge f_4 + f_2 \wedge f_3 &\mapsto f_1 \wedge f_3 + f_4 \wedge f_2, \end{aligned}$$

and meanwhile interchanges anti-self-dual forms likewise.

It follows that when we replace f_2 by its negative, and interchange the pair f_3 and f_4 and the pair e_1 and e_2 , we will preserve the same type of decomposition except for a possible sign change and a permutation of the last two diagonal entries of D_1 and D_2 . Note that under the transformation

$$\begin{aligned} e_1 &\longleftrightarrow e_2, & e_3 &\rightarrow e_3, \\ f_3 &\longleftrightarrow f_4, & f_1 &\rightarrow f_1, & f_2 &\rightarrow -f_2, \end{aligned} \tag{33}$$

the following data are exchanged:

$$\begin{aligned} \alpha &\longleftrightarrow \mu, & \gamma &\longleftrightarrow -\nu, & F_{11,22} &\longleftrightarrow F_{11,22}, & F_{14,23} &\longleftrightarrow F_{14,23}, \\ F_{14,21} &\longleftrightarrow F_{23,11} = -F_{14,22} \end{aligned} \tag{34}$$

Therefore, whenever an identity involving only these quantities hold, the transformed identity via (34) must hold as well.

The swapping produces two different representatives of the same moduli point.

Proposition 2. *Assume $\beta = 0$. If $F_{14,21}F_{14,22} \neq 0$, then we have*

$$\alpha = \gamma, \quad \mu = -\nu, \quad F_{11,22} = F_{14,23}.$$

It is part of the grand moduli $\mathcal{G}_{(3,4,8)}$ when we set $\beta = 0$.

If $F_{14,21} \neq 0$ and $F_{14,22} = 0$, then

$$\mu = -\nu, \quad F_{11,22} = F_{14,23}.$$

Furthermore, if $\alpha = \gamma$, then it is part of the grand moduli (31) in which we set $\beta = F_{14,22} = 0$, depending on three parameters. On the other hand, if $\alpha \neq \gamma$, then these orthogonal multiplications are governed by (43) and (44) below and depends on the three parameters α, μ , and γ .

The case when $F_{14,21} = 0$ and $F_{14,22} \neq 0$ is equivalent to the preceding one via (34).

In fact, there is a one-to-one correspondence between the case in which $\beta \neq 0$ and $F_{14,21} = F_{14,22} = 0$ in Proposition 1, and the above case in which $\beta = 0$ and $F_{14,21} \neq 0$ and $F_{14,22} = 0$ (or $F_{14,21} = 0$ and $F_{14,22} \neq 0$).

Proof. By (17), we have $\sigma_1\sigma_2 \neq 0$. Since the rows of c_2 are orthogonal, its first and second rows give

$$F_{14,21}F_{14,22}\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_2^2}\right) = 0,$$

so that $\sigma_1 = \sigma_2$, or $\mu^2 = \nu^2$, if $F_{14,21}F_{14,22} \neq 0$; by a coordinate sign change we may assume $\mu = -\nu$. Moreover, with $\mu = -\nu$, the second and the fourth rows of c_2 , being orthogonal, yields

$$F_{14,21}(F_{11,22} - F_{14,23}) = 0,$$

so that $F_{11,22} = F_{14,23}$ when $F_{14,21}F_{14,22} \neq 0$; the second and third rows then give

$$F_{14,22} \mu(\alpha - \gamma) = 0,$$

from which we conclude

$$\alpha = \gamma, \quad \text{under the condition } \mu \neq 0 \text{ and } F_{14,21}F_{14,22} \neq 0.$$

Continue to assume $F_{14,21}F_{14,22} \neq 0$. If $\mu = 0$ then $\mu = \nu = 0$ and so $\sigma_1 = \sigma_2 = 1$. The lengths of the the first two rows of F_2 being 1 implies

$$F_{11,22}^2 + F_{14,21}^2 + F_{14,22}^2 + \alpha^2 = 1, \quad F_{11,22}^2 + F_{14,21}^2 + F_{14,22}^2 + \gamma^2 = 1, \quad (35)$$

so that we obtain $\alpha^2 = \beta^2$. After a coordinate sign change, we may assume $\alpha = \gamma$ since $\mu = -\nu = 0$ is not affected.

If $F_{14,22} = 0$ and $F_{14,21} \neq 0$, we first establish $\mu^2 = \nu^2$.

The orthogonality of the first and the third rows, and respectively the second and the fourth rows of c_2 give, after cancelling $F_{14,21}$,

$$\begin{aligned} \frac{1}{\sigma_2^2}F_{11,22} - \frac{1}{\sigma_1^2}F_{14,23} &= \alpha\left(\frac{\mu}{\sigma_1^2} + \frac{\nu}{\sigma_2^2}\right), \\ \frac{1}{\sigma_1^2}F_{11,22} - \frac{1}{\sigma_2^2}F_{14,23} &= -\gamma\left(\frac{\mu}{\sigma_1^2} + \frac{\nu}{\sigma_2^2}\right), \end{aligned} \quad (36)$$

from which we solve to obtain

$$\begin{aligned} \left(\frac{\sigma_1^2}{\sigma_2^2} - \frac{\sigma_2^2}{\sigma_1^2}\right)F_{11,22} &= (\alpha\sigma_1^2 + \gamma\sigma_2^2)\left(\frac{\mu}{\sigma_1^2} + \frac{\nu}{\sigma_2^2}\right), \\ \left(\frac{\sigma_1^2}{\sigma_2^2} - \frac{\sigma_2^2}{\sigma_1^2}\right)F_{14,23} &= (\alpha\sigma_2^2 + \gamma\sigma_1^2)\left(\frac{\mu}{\sigma_1^2} + \frac{\nu}{\sigma_2^2}\right). \end{aligned} \quad (37)$$

Since the rows of F_2 are of unit length, we have

$$\begin{aligned} \frac{\sigma_1^2}{\sigma_2^2}(F_{11,22} - \alpha\nu)^2 - \sigma_1^2(1 - \alpha^2) &= -F_{14,21}^2, \\ \frac{\sigma_2^2}{\sigma_1^2}(F_{11,22} + \gamma\mu)^2 - \sigma_2^2(1 - \gamma^2) &= -F_{14,21}^2, \\ \frac{\sigma_1^2}{\sigma_2^2}(F_{14,23} - \gamma\nu)^2 - \sigma_1^2(1 - \gamma^2) &= -F_{14,21}^2, \\ \frac{\sigma_2^2}{\sigma_1^2}(F_{14,23} + \alpha\mu)^2 - \sigma_2^2(1 - \alpha^2) &= -F_{14,21}^2. \end{aligned} \quad (38)$$

Substituting (37) into (38) and cancelling $F_{14,21}^2$ by subtraction, we derive, respectively, for the first and second pairs in (38)

$$\begin{aligned} ((1 - \alpha^2)\sigma_1^2 - (1 - \gamma^2)\sigma_2^2) \left(\frac{\sigma_1^2}{\sigma_2^2} - \frac{\sigma_2^2}{\sigma_1^2} \right)^2 &= \left(\frac{\sigma_1^2}{\sigma_2^2} - \frac{\sigma_2^2}{\sigma_1^2} \right) (\alpha^2 - \gamma^2) (\mu^2 - \nu^2), \\ ((1 - \alpha^2)\sigma_2^2 - (1 - \gamma^2)\sigma_1^2) \left(\frac{\sigma_1^2}{\sigma_2^2} - \frac{\sigma_2^2}{\sigma_1^2} \right)^2 &= \left(\frac{\sigma_1^2}{\sigma_2^2} - \frac{\sigma_2^2}{\sigma_1^2} \right) (\alpha^2 - \gamma^2) (\mu^2 - \nu^2). \end{aligned} \quad (39)$$

Suppose $\sigma_1 \neq \sigma_2$. We equate the left hand sides and cancel the common fraction to see

$$(2 - \alpha^2 - \gamma^2)\sigma_1^2 = (2 - \alpha^2 - \gamma^2)\sigma_2^2,$$

which gives

$$2 = \alpha^2 + \gamma^2,$$

so that in fact

$$\alpha^2 = \gamma^2 = 1,$$

which implies that the first row of c_2 is identically zero, so that $F_{14,21} = 0$, a contradiction. Thus $\sigma_1 = \sigma_2$, i.e., $\mu^2 = \nu^2$, so that after a coordinate sign change we may assume

$$\mu = -\nu. \quad (40)$$

It follows from (36) that

$$F_{11,22} = F_{14,23}. \quad (41)$$

Meanwhile, since each row of F_2 is of unit length, we calculate the first and the second to see

$$2\alpha\mu F_{11,22} + (F_{11,22}^2 + F_{14,21}^2) = \sigma^2 - \alpha^2, \quad 2\gamma\mu F_{11,22} + (F_{11,22}^2 + F_{14,21}^2) = \sigma^2 - \gamma^2, \quad (42)$$

with $\sigma^2 = 1 - \mu^2$. Cancelling out $F_{11,22}^2 + F_{14,21}^2$ we obtain

$$-2(\alpha - \gamma)\mu F_{11,22} = \alpha^2 - \gamma^2.$$

In particular, if $\alpha \neq \gamma$, then

$$-2\mu F_{11,22} = \alpha + \gamma. \quad (43)$$

Substituting it into the sum of the two identities in (42) we obtain

$$F_{14,21}^2 + F_{11,22}^2 = \sigma^2 + \alpha\gamma. \quad (44)$$

There are indeed orthogonal multiplications in this category for which $\alpha \neq \gamma$. For instance, let us assume $\alpha = 0$ and $\gamma \neq 0$. Then (43) and (44) assert that $F_{14,21}$ exists so long as

$$\sigma^2 \geq \frac{\gamma^2}{4\mu^2}, \quad \text{or} \quad 4\mu^2(1 - \mu^2) \geq \gamma^2,$$

for which there are γ once μ is chosen appropriately.

Now, observe that, similar to (33), when we apply the orthogonal element in $SO(4)$ that maps f_1 and $-f_4$, f_4 to f_1 and fixes both f_2 and f_3 , we

interchange the first two columns of both the self-dual and anti-self-dual parts of the Parker matrix such that

$$\begin{aligned} f_1 \wedge f_2 + f_3 \wedge f_4 &\rightarrow -(f_1 \wedge f_3 + f_4 \wedge f_2) \\ f_1 \wedge f_3 + f_4 \wedge f_2 &\rightarrow f_1 \wedge f_2 + f_3 \wedge f_4, \\ f_1 \wedge f_4 + f_2 \wedge f_3 &\rightarrow f_1 \wedge f_4 + f_2 \wedge f_3, \end{aligned}$$

and likewise for anti-self-dual forms. Moreover, we interchange e_2 and e_3 while fixing e_1 , so that the transformation

$$\begin{aligned} e_1 &\rightarrow e_1, & e_2 &\rightarrow e_3, & e_3 &\rightarrow e_2, \\ f_1 &\rightarrow -f_4, & f_4 &\rightarrow f_1, & f_2 &\rightarrow f_2, & f_3 &\rightarrow f_3, \end{aligned} \tag{45}$$

retains the diagonal-and-upper-triangular pattern. With this, the case when $\beta = 0 = F_{14,22}$ and $F_{14,21} \neq 0$ is converted to the case where $\beta \neq 0$ and $F_{14,21} = F_{14,22} = 0$ given in Proposition 1.

This is the symmetry of the moduli we will explore next. We denote the resulting F -quantities obtained through the transformation (45) with an extra $*$ to avoid confusion. We have

$$\begin{aligned} \mu^* &= F_{24,31}^* = -F_{31,24} = -\mu, & \nu^* &= F_{23,32}^* = F_{33,22} = -\nu, \\ \beta^* &= F_{31,14}^* = -F_{24,11} = F_{14,21}, & \alpha^* &= F_{31,13}^* = -F_{24,13} = F_{14,23}, \\ \gamma^* &= F_{22,11} = F_{11,22}, & F_{11,22}^* &= -F_{14,32} = -\gamma, & F_{14,23}^* &= F_{11,33} = -\alpha. \end{aligned} \tag{46}$$

If we invoke Proposition 1, where $\mu^* = -\nu^*$ and $\alpha^* = \gamma^*$, we again conclude (40) and (41) obtained through algebraic means. Meanwhile, the class

$$F_{14,23}^* = -F_{11,22}^* - 2\alpha^* \mu^*$$

in (32) now translates into

$$-\alpha = \gamma + 2\mu F_{14,23},$$

which is exactly (43) in view of (44). \square

Corollary 1. *The case when $\beta = 0 = F_{14,22}$ and $F_{14,21} \neq 0$ (or $\beta = 0 = F_{14,21}$ and $F_{14,22} \neq 0$) is equivalent to the generic case when $\beta \neq 0$ and $F_{14,21} = F_{14,22} = 0$ in the grand moduli $\mathcal{G}_{(3,4,8)}$. Therefore, the latter are boundary points of the coarse fundamental domain identified with the former to belong to the grand moduli.*

Proof. This is through the symmetry (45) in the preceding proof. \square

Remark 1. *Note that in the grand moduli, $c_1 = \sigma Id$, c_2 is of the form*

$$c_2 = -\frac{\beta\mu}{\sigma} Id + M,$$

where M is skew-symmetric since $\alpha = \gamma$ and $\mu = -\nu$. The Hurwitz condition

$$c_2 c_2^{tr} + w_2 w_2^{tr} = Id$$

is reduced to

$$MM^{tr} = \theta^2 Id, \quad \theta = \sqrt{1 - \frac{\beta^2 \mu^2}{\sigma^2} - \alpha^2}, \quad \sigma = \sqrt{1 - \mu^2}.$$

Meanwhile, w_1 and w_2 are of the forms

$$w_1 = \mu J, \quad w_2 = \tau L, \quad \tau = \sqrt{\alpha^2 + \beta^2},$$

where J, L are skew-symmetric and orthogonal. Therefore, matrices in the grand moduli are, up to adjoint equivalence, of the form

$$F_1 = (\sigma Id \quad \mu J), \quad F_2 = \left(-\frac{\beta\mu}{\sigma} Id + \theta M \quad \tau L\right), \quad F_3 = (0 \quad Id),$$

satisfying

$$\mu\tau(JL + LJ) = 2\beta\mu Id.$$

4.4. The degenerate case $\beta = F_{14,21} = F_{14,22} = 0$. This is the case when the upper triangular T in (11) is also diagonal. It includes the case when either σ_1 or σ_2 is zero as given in (17).

It is more convenient to denote

$$\alpha = \cos(\phi), \quad \gamma = \cos(\psi), \quad \mu = \cos(\theta), \quad \nu = \cos(\eta), \quad 0 \leq \phi, \psi, \theta, \eta \leq \pi.$$

Then we have $\sigma_1 = \sin(\theta)$, $\sigma_2 = \sin(\eta)$, and c_2 in (16) is

$$c_2 = \begin{pmatrix} 0 & \pm \sin(\phi) & 0 & 0 \\ \pm \sin(\psi) & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm \sin(\phi) \\ 0 & 0 & \pm \sin(\psi) & 0 \end{pmatrix},$$

with

$$\begin{aligned} \frac{F_{11,22} - \alpha\nu}{\sin(\eta)} &= \pm \sin(\phi), & \frac{-F_{11,22} - \gamma\mu}{\sin(\theta)} &= \pm \sin(\psi), \\ \frac{-F_{14,23} - \alpha\mu}{\sin(\theta)} &= \pm \sin(\phi), & \frac{F_{14,23} - \gamma\nu}{\sin(\eta)} &= \pm \sin(\psi). \end{aligned} \quad (47)$$

We obtain

$$F_{11,22} = \cos(\phi \pm \eta) = -\cos(\psi \pm \theta), \quad F_{14,23} = \cos(\psi \pm \eta) = -\cos(\phi \pm \theta). \quad (48)$$

This category is where a large set of anomaly occurs for which $\alpha^2 \neq \gamma^2$ and $\mu^2 \neq \nu^2$.

The orthogonal multiplications in this category that satisfy $\alpha = \gamma$ and $\mu = -\nu$ to belong to the grand moduli is when

$$\cos(\phi) = \cos(\psi), \quad \cos(\theta) = -\cos(\eta), \quad (49)$$

so that

$$\phi = \psi, \quad \theta + \eta = \pi$$

with the constraint

$$\cos(\phi \pm \eta) = -\cos(\phi \pm \theta).$$

More generally, the condition $\alpha^2 = \gamma^2$ and $\mu^2 = \nu^2$ is equivalent to

$$\sin(\eta) = \sin(\theta), \quad \sin(\phi) = \sin(\psi) \quad (50)$$

with the constraint (48).

A family that does not satisfy (50) is when $\theta + \eta + \phi + \psi = \pi$.

Putting Corollary 1 and the discussion in this section, we obtain the following.

Corollary 2. *The moduli space of orthogonal multiplications of type $[3, 4, 8]$ consists of two components. The 5-dimensional grand moduli $\mathcal{G}_{(3,4,8)}$ given in Proposition 1 with the constraints (29), (30), (31), and the 3-dimensional degenerate moduli $\mathcal{X}_{(3,4,8)}$ given in (47) with the constraint (48). They intersect at the points where (49) holds.*

Remark 2. *Corollary 2 is reminiscent of the Cartan Umbrella $z(x^2 + y^2) = x^3$. It is a real irreducible variety for which the umbrella canopy is 2-dimensional, similar to the grand moduli $\mathcal{G}_{(3,4,8)}$, and the umbrella shaft is the 1-dimensional z axis, similar to the anomalous $\mathcal{X}_{(3,4,8)}$. A property on the canopy need not hold on the shaft.*

4.5. Type $[3, 4, 7]$. As a consequence of Section 4.4, when the orthogonal multiplication is of type $[3, 4, 7]$, we know $\sigma_1\sigma_2 = 0$, because otherwise, c_1 and w_3 would account for 8 dimensions, not 7. It follows by (17) that the orthogonal multiplication belongs to the degenerate case in Section 4.4.

We may assume $\sigma_1 = 0$ and $\sigma_2 \neq 0$ up to range equivalence, i.e., $\mu^2 = 1$ and $\nu^2 < 1$, then already c_1 and w_3 account for 6 dimensions, so that either the first or the fourth column of c_2 is zero for the range dimension to be 7. We may assume it is the first column that vanishes up to range equivalence, i.e., $\sin(\psi) = 0$, or, $\gamma^2 = 1$. The Hurwitz condition dictates

$$\pm \sin(\phi) = -\alpha\nu - \gamma\mu, \quad \alpha\mu + \gamma\nu = 0,$$

from which we conclude

$$\sin(\phi) = \sigma_2, \quad \mu^2 = \gamma^2 = 1, \quad \alpha^2 = \nu^2, \quad \sigma_2 = \sqrt{1 - \nu^2}.$$

The moduli dimension of such orthogonal multiplications is 1.

5. A MORE RIGID RANGE EQUIVALENCE AND ITS APPLICATION TO ISOPARAMETRIC HYPERSURFACES

Recall the process leading to (15) and (16). Once we normalize the Parker matrix as given in (11), we look at the span V of $e_3 \circ f_1, \dots, e_3 \circ f_4$ and its orthogonal complement V^\perp , relative to which we set the anchor matrix F_3 to be $(0 \quad Id)$. From F_3 we can set F_1 in the canonical form (15) in agreement with orthogonal multiplications of type $[2, 4, 8]$ in Section 2, and consequently build F_2 in terms of the data (13) arising from the Parker matrix normalization.

Consider the case when the range equivalence is more rigidly restricted to a fixed decomposition $\mathbb{R}^8 = \mathbb{R}^4 \oplus \mathbb{R}^4$, so that only $SO(4) \oplus SO(4) \subset SO(8)$ is allowed. Accordingly, an orthogonal multiplication F in the coarse fundamental domain, with $F_3 = (0 \quad Id)$ obtained by the $SO(8)$ range equivalence,

may turn into one for which the first block of F_3 is nonzero when imposing the more rigid $SO(4) \oplus SO(4)$ range equivalence. That is, V need not be the prescribed second copy of \mathbb{R}^4 , even when F_1 relative to the fixed decomposition is the prescribed form as in (15).

As an example, consider the generic case in Proposition 1. We know $|\mu| = |\nu| < 1$ by Lemma 1 so that $\sigma = \sigma_1 = \sigma_2 > 0$. Set $\lambda := |\mu|/\sigma$ and $J := w_1/\sigma$ with w_1 given in (15). Multiplying the orthogonal matrix

$$O := \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A = I - JC, \quad B = J - JD, \quad C = DJ - J, \\ D = -D^{tr}, \quad DD^{tr} = (1 - \lambda^2)/(1 + \lambda^2) Id$$

on the right of F_1, F_2, F_3 in (15) and (16), we see

$$F_1^* := F_1 O = F, \quad F_2^* := F_2 O, \quad F_3^* := \begin{pmatrix} C & D \end{pmatrix}.$$

That is, F_1^* of the orthogonal multiplication F^* , relative to the fixed decomposition of \mathbb{R}^8 , assumes the same prescribed form as given in (15). However, V , the span of the rows of F_3^* , is not in general the second copy of \mathbb{R}^4 in the fixed decomposition of \mathbb{R}^8 ; performing basis change over each \mathbb{R}^4 summand, which amounts to multiplying on the right of O by an orthogonal matrix in the diagonal block form, does not convert O to the identity matrix in general.

In a similar fashion, given a degenerate orthogonal multiplication F representing $\mathcal{X}_{(3,4,8)}$ in the coarse fundamental domain, assume that F_1 is as prescribed in (15) relative to the fixed decomposition of \mathbb{R}^8 . To determine that V is the fixed second copy of \mathbb{R}^4 , we now choose F_1 as the anchor matrix in place of F_3 to utilize the existing symmetry. The data in (51) below then dictates that V is the fixed second copy of \mathbb{R}^4 . Meanwhile, F_2 is now represented in two coarse fundamental domains of $\mathcal{X}_{(3,4,8)}$, one for which F_3 is the anchor matrix and the other for which F_1 is. Thus, it is only when the orthogonal multiplication belongs to the intersection of these two coarse fundamental domains can we assert that V is the fixed second copy of \mathbb{R}^4 . We carry out the details as follows.

Corollary 3. *Notations and conditions as above, assume $\sigma_1\sigma_2 \neq 0$ for the degenerate case presented in Section 4.4. Then V is the fixed second copy of \mathbb{R}^4 only when the orthogonal multiplication belongs to the grand moduli $\mathcal{G}_{(3,4,8)}$.*

Proof. We utilize the symmetry (45) explicitly, letting F_1 be the ‘‘anchor matrix’’ instead so that $F_1 = (0 \quad Id)$. We calculate to see, relative to F_1 ,

$$F_3 = (c_3 \quad w_3), \quad c_3 = \begin{pmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \sigma_2 & 0 \\ 0 & 0 & 0 & \sigma_1 \end{pmatrix}, \quad w_3 = \begin{pmatrix} 0 & 0 & 0 & \mu \\ 0 & 0 & \nu & 0 \\ 0 & -\nu & 0 & 0 \\ -\mu & 0 & 0 & 0 \end{pmatrix}, \quad (51)$$

and $F_2 = (c_2 \ w_2)$, where

$$c_2 = \begin{pmatrix} F_{14,21}\mu/\sigma_1 & 0 & (F_{11,22} - \alpha\nu)\nu/\sigma_2 & 0 \\ 0 & -F_{14,21}\nu/\sigma_2 & 0 & -(F_{11,22} + \gamma\mu)\mu/\sigma_1 \\ (F_{14,23} + \alpha\mu)\mu/\sigma_1 & 0 & -F_{14,21}\nu/\sigma_2 & 0 \\ 0 & -(F_{14,23} - \gamma\nu)\nu/\sigma_2 & 0 & F_{14,21}\mu/\sigma_1 \end{pmatrix},$$

$$w_2 = \begin{pmatrix} 0 & F_{11,22} & 0 & -F_{14,21} \\ -F_{11,22} & 0 & F_{14,21} & 0 \\ 0 & -F_{14,21} & 0 & -F_{14,23} \\ F_{14,21} & 0 & F_{14,23} & 0 \end{pmatrix}.$$

(45) tells us to interchange the first and fourth columns and rows, we end up with

$$c_3 = \begin{pmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \sigma_2 & 0 \\ 0 & 0 & 0 & \sigma_1 \end{pmatrix}, \quad w_3 = \begin{pmatrix} 0 & 0 & 0 & -\mu \\ 0 & 0 & \nu & 0 \\ 0 & -\nu & 0 & 0 \\ \mu & 0 & 0 & 0 \end{pmatrix}, \quad (52)$$

$$c_2 = \begin{pmatrix} F_{14,21}\mu/\sigma_1 & -(F_{14,23} - \gamma\nu)\nu/\sigma_2 & 0 & 0 \\ -(F_{11,22} + \gamma\mu)\mu/\sigma_1 & -F_{14,21}\nu/\sigma_2 & 0 & 0 \\ 0 & 0 & -F_{14,21}\nu/\sigma_2 & (F_{14,23} + \alpha\mu)\mu/\sigma_1 \\ 0 & 0 & (F_{11,22} - \alpha\nu)\nu/\sigma_2 & F_{14,21}\mu/\sigma_1 \end{pmatrix},$$

$$w_2 = \begin{pmatrix} 0 & 0 & F_{14,23} & F_{14,21} \\ 0 & 0 & F_{14,21} & -F_{11,22} \\ -F_{14,23} & -F_{14,21} & 0 & 0 \\ -F_{14,21} & F_{11,22} & 0 & 0 \end{pmatrix}. \quad (53)$$

Comparing (52) and (53) with (15) and (16), upon which we need to change the sign of both the first column and row in (52) and (53), dictated by (45), to make the \pm signs of the two sets of c - and w -matrices agreeable, we conclude, where we denote the new F -quantities with an extra $*$, that we obtain the same identities,

$$\mu^* = -\mu, \quad \nu^* = -\nu, \quad \beta^* = F_{14,21}, \quad \alpha^* = F_{14,23}, \quad \gamma^* = F_{11,22}$$

as in (46). Furthermore, we have

$$\begin{aligned} F_{11,22}^* - \alpha^* \nu^* &= -(F_{14,23} - \gamma\nu)\nu, & F_{11,22}^* + \gamma^* \nu^* &= (F_{11,22} + \gamma\mu)\mu \\ -F_{14,23}^* - \alpha^* \mu^* &= (F_{14,23} + \alpha\mu)\mu, & F_{14,23}^* - \gamma^* \nu^* &= (F_{11,22} - \alpha\nu)\nu. \end{aligned} \quad (54)$$

Incorporating (46) and (54), we see

$$\gamma = \gamma\nu^2, \quad \gamma = \gamma\mu^2, \quad \alpha = \alpha\nu^2, \quad \alpha = \alpha\mu^2. \quad (55)$$

If either α or γ is nonzero, then $\mu^2 = \nu^2 = 1$ so that $\sigma_1 = \sigma_2 = 0$, contradicting $\sigma_1\sigma_2 \neq 0$.

Otherwise, $\alpha = \gamma = 0$. The rows of c_2 in (16), being of unit length, then gives $\sigma_1 = \sigma_2$, or, $\mu^2 = \nu^2$, so that we may assume $\mu = -\nu$ without affecting $\alpha = \beta = 0$. In particular, c_2 in (47) is skew-symmetric and nonzero. \square

Remark 3. *Although the category where $\beta = 0 = F_{14,22}$ and $F_{14,21} \neq 0$ can be interchangeably converted to the category where $\beta \neq 0$ and $F_{14,21} = F_{14,22} = 0$ that lives in the grand moduli, we can also see directly that the conclusion of Corollary 3 holds as well for both categories as follows.*

Assume $\beta = 0 = F_{14,22}$ and $F_{14,21} \neq 0$. We know $\mu = -\nu$ by (40), so that we slightly modify (55) to conclude

$$\gamma = \gamma\mu^2, \quad \alpha = \alpha\mu^2.$$

Now $\mu^2 \neq 1$, since otherwise $\sigma = 0$ so that $F_{14,21} = 0$ by (17), which is absurd. Thus, $\alpha = \gamma = 0$. But then the rows of c_2 in (16), being of unit length, implies that $\mu^2 = \nu^2$, so that we may assume $\mu = -\nu$ without affecting $\alpha = \gamma = 0$. In particular, c_2 in (16) is skew-symmetric and nonzero.

On the other hand, assume $\beta \neq 0$ and $F_{14,21} = F_{14,22} = 0$. Going through the same arguments we obtain $\alpha = \alpha\mu^2$ since we know $\alpha = \gamma$ and $\mu = -\nu$ by Proposition 1, so that $\alpha = 0 = \gamma$ and so $F_{11,22}^2 = F_{14,23}^2$ by (32). Note that in this case, c_2 in (53) is slightly modified with $-\beta\mu^2/\sigma, \beta\nu^2/\sigma, -\beta\nu^2/\sigma, \beta\mu^2/\sigma$ filling the (1, 4)-, (2, 3)-, (3, 2)-, and (4, 1)-entries, respectively, where $\sigma = \sigma_1 = \sigma_2$. Suppose $F_{11,22} = F_{14,23} = 0$, then the modified (53) implies $\beta\mu^2 = \pm\sigma$ since each row of F_2 is of unit length and, except for one entry, all other entries of its first row are zero; however, (16) implies $\beta^2 = \sigma^2$ for the same reason, so that $\mu^2 = 1$, which is absurd as then $0 = \sigma = \beta \neq 0$. Hence, $F_{11,22}^2 = F_{14,23}^2$ are nonzero. In particular, c_2 in (16) is of the form $aId + C$ for some real number a and some nonzero skew-symmetric C .

To make a long story short, let us remark that orthogonal multiplications of type $[3, 4, 8]$ play a decisive role in the classification of isoparametric hypersurfaces with four principal curvatures and multiplicity pair $(7, 8)$ in S^{31} [8, Section 7]. The $SO(4) \oplus SO(4)$ range equivalence is important in [8, Section 7] because the orthogonal multiplication must respect a prescribed $\mathbb{R}^4 \oplus \mathbb{R}^4$ decomposition intrinsic to the underlying isoparametric structure that is associated with the focal manifolds of the hypersurface, in such a way that, relative to this intrinsic decomposition, F_1 is prescribed as in (15) and the span of the rows of F_3 is the second copy of \mathbb{R}^4 . What is developed in the preceding section and this section asserts, in the isoparametric situation, where necessarily $\sigma_1\sigma_2 \neq 0$, that $\alpha = \gamma$ and $\mu = -\nu$ uniformly, and, moreover, c_2 in (16) is of the form

$$c_2 = aId + C, \quad a \in \mathbb{R}, \quad 0 \neq C \text{ is skew-symmetric,}$$

so that, up to adjoint equivalence,

$$c_2 = aId + b \begin{pmatrix} I & 0 \\ 0 & \pm I \end{pmatrix}, \quad I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad b \neq 0,$$

which, indicated in an *ad hoc* fashion in [8], is pivotal for establishing the decisive Corollary 7.3 there.

6. MODULI SPACE OF TYPE $[3, 4, p], p \leq 12$

A full orthogonal multiplication of type $[3, 4, 12]$ has a 9-dimensional moduli in $\wedge^2 \mathbb{R}^3 \otimes \wedge^2 \mathbb{R}^4 / SO(3) \otimes SO(4)$. We can see this explicitly. Namely, the process to carry out (15) and (16) goes through verbatim. Next, we complete the orthonormal basis of \mathbb{R}^{12} from u_1, \dots, u_8 constructed in Section 4.1 by augmenting four basis vectors $u_{-4}, u_{-3}, u_{-2}, u_{-1}$ such that

$$\tilde{F}_3 = \begin{pmatrix} 0 & 0 & Id \end{pmatrix}, \quad \tilde{F}_1 = \begin{pmatrix} 0 & c_1 & w_1 \end{pmatrix}, \quad \tilde{F}_2 = \begin{pmatrix} \epsilon_2 & c_2 & w_2 \end{pmatrix}, \quad (56)$$

with c_1, c_2, w_1, w_2 given in (16).

It is more convenient to use the normal exponential map to parametrize \tilde{F}_2 . Namely, we let the \mathbb{R}^8 containing the rows of $(c_2 \ w_2)$ be horizontal so that the \mathbb{R}^4 containing the rows of ϵ_2 is vertical. For any vector v , identified with a vector in the vertical \mathbb{R}^4 , normal to the horizontal S^7 at each of its point x , we have the map

$$Exp : (x, v) \mapsto \cos(|v|) x + \sin(|v|) v/|v|, \quad (p, 0) \mapsto p, \quad (57)$$

which is a diffeomorphism from $S^7 \times D$ into S^{11} describing a tubular neighborhood of S^7 in S^{11} , where D is a sufficiently small disk around 0 in \mathbb{R}^4 . In light of this, the j th row of \tilde{F}_2 can be written as follows. In (56), let z_j be the j th row of $(c_2 \ w_2)$ and y_j be the j th row of ϵ_2 . Let

$$x_j := z_j/|z_j|, \quad \sin(\omega_j) := |y_j|, \quad v_j := \omega_j y_j/|y_j|.$$

Then

$$Exp(x_j, v_j) = (y_j, z_j) = j\text{th row of } \tilde{F}_2.$$

We have the map

$$\Pi : \tilde{F}_2 \mapsto (|\pi \circ Exp^{-1}((y_1, z_1))|, \dots, |\pi \circ Exp^{-1}((y_1, z_1))|) = (\omega_1, \omega_2, \omega_3, \omega_4), \quad (58)$$

where $\pi : (x, v) \mapsto v$. Note that the angles ω_j , though set to be nonnegative in (57), can in fact be extended to negative values since

$$Exp(p, -v) = \cos(-|v|) p + \sin(-|v|) v/|v|,$$

which we adopt henceforth.

The moduli of type $[3, 4, 12]$ is parametrized by the nine generic variables

$$\alpha, \beta, \gamma, \mu, \nu, F_{14,21}, F_{14,22}, F_{11,22}, F_{14,23}. \quad (59)$$

Let \mathcal{V} be the span of the rows of $\tilde{F}_1, \tilde{F}_2, \tilde{F}_3$. We have

$$p = \dim(\mathcal{V}).$$

Suppose $\sigma_1 \sigma_2 \neq 0$. Then it is expected that generically $p = 12 - n$, where n is the number of zeros of v_1, \dots, v_4 . We studied extensively the important case when $n = 4$ in the previous sections, while $n = 0$ gives the dimension of the entire moduli. The expectation is indeed the case. We next show that there is a moduli of type $[3, 4, 12 - n]$ for each $1 \leq n \leq 3$ under the generic assumption that $\sigma_1 \sigma_2 \neq 0$. To this end, observe that in the degenerate case

$\beta = F_{14,21} = F_{14,22} = 0$, we have, as in Section 4.4, $\tilde{F}_2 = (\epsilon_2 \ c_2 \ w_2)$, where

$$(\epsilon_2 \ c_2) = \begin{pmatrix} s \sin(\phi) & 0 & 0 & 0 & 0 & s \cos(\phi) & 0 & 0 \\ 0 & t \sin(\psi) & 0 & 0 & t \cos(\psi) & 0 & 0 & 0 \\ 0 & 0 & s \sin(\zeta) & 0 & 0 & 0 & 0 & s \cos(\zeta) \\ 0 & 0 & 0 & t \sin(\xi) & 0 & 0 & t \cos(\xi) & 0 \end{pmatrix}. \quad (60)$$

for some angles ϕ, ψ, ζ, ξ between 0 and π , where

$$s = \sqrt{1 - \alpha^2}, \quad t = \sqrt{1 - \gamma^2}, \quad (61)$$

w_2 is given as in (16). The Hurwitz condition says

$$t \sigma_1 \cos(\psi) + s \sigma_2 \cos(\phi) = -\alpha\nu - \gamma\mu, \quad s \sigma_1 \cos(\zeta) + t \sigma_2 \cos(\xi) = -\alpha\mu - \gamma\nu. \quad (62)$$

If we set $\phi = 0$ and the other three angles equal to $\pi/2$, then the Hurwitz condition is reduced to

$$s\sigma_2 = -\alpha\nu - \gamma\mu, \quad \alpha\mu + \gamma\nu = 0,$$

which is equivalent to

$$\sqrt{1 - \alpha^2} \sqrt{1 - \nu^2} = (\gamma^2 - \alpha^2)\nu/\alpha. \quad (63)$$

The equation does carry solutions; for instance, one can set $\nu = \epsilon\alpha$. Then, we are solving

$$\epsilon^2 \gamma^4 - 2\epsilon^2 \alpha^2 \gamma^2 + (1 + \epsilon^2)\alpha^2 - 1 = 0, \quad (64)$$

which has a solution

$$\gamma^2 = \alpha^2 + s \sqrt{\frac{1}{\epsilon^2} - \alpha^2}$$

for $0 < \gamma^2 < 1$ as long as we choose ϵ such that

$$1 < \epsilon^2 < 1/\alpha^2, \quad \epsilon > 0,$$

The upshot is that we have now $p = 11$ under the generic assumption that $\sigma_1 \sigma_2 \neq 0$.

To prove the generic moduli dimension of type $[3, 4, 11]$ is 8, observe that we can perturb slightly the angle ϕ away from zero, then (64) is perturbed into

$$\epsilon^2 \gamma^4 - 2\epsilon^2 \alpha^2 \gamma^2 + (1 + \epsilon^2)\alpha^2 - 1 = s^2 \sigma_2^2 \sin^2(\phi),$$

so that we obtain

$$\gamma^2 = \alpha^2 + s \sqrt{\frac{1}{\epsilon^2} - \alpha^2 + \sigma_2^2 \sin^2(\phi)}, \quad (65)$$

and so we have solutions for γ for sufficiently small ϕ . It follows that the analytic map

$$\Pi_1 : \tilde{F}_2 \mapsto \omega_1,$$

where ω_1 is defined in (58), from \mathbb{R}^9 to \mathbb{R} is surjective around the image point 0 (we allow negative ϕ). Moreover, since ϕ and s are independent variables in (65), it is then clear that 0 is a regular value of Π_1 . Hence, by the rank theorem $\Pi_1^{-1}(0)$ is of dimension 8.

Similarly, we can let $\phi = \psi = 0$ and the other two angles equal $\pi/2$. Then the Hurwitz condition reads

$$t\sigma_1 + s\sigma_2 = -\alpha\nu - \gamma\mu, \quad \alpha\mu + \gamma\nu = 0.$$

Solving μ from the second equation and substituting it into the first yields

$$\sqrt{1 - \mu^2} = \frac{-(\gamma^2 - \alpha^2)\nu/\alpha - \sqrt{1 - \alpha^2}\sqrt{1 - \nu^2}}{\sqrt{1 - \gamma^2}}. \quad (66)$$

There is a solution for $0 < \mu^2 < 1$ if we choose appropriately α and ν close to 1 and γ close to zero. So, $p = 10$ under the generic condition that $\sigma_1\sigma_2 \neq 0$.

Similar to the preceding case, (66) is perturbed to

$$\sqrt{1 - \mu^2} = \frac{-(\gamma^2 - \alpha^2)\nu/\alpha - \sqrt{1 - \alpha^2}\sqrt{1 - \nu^2} \cos(\phi)}{\sqrt{1 - \gamma^2} \cos(\psi)}$$

with solution for sufficiently small ϕ and ψ . Once more, since s, t, ϕ, ψ are independent variables and we allow negative angles, the map $\Pi_2 : \tilde{F}_2 \mapsto (\omega_1, \omega_2)$ assumes $(0, 0)$ as a regular value, so that $\Pi_2^{-1}((0, 0))$ is of dimension 7.

In the same way, when we let $\phi = \psi = \zeta = 0$ and $\xi = \pi/2$, the Hurwitz condition reads

$$t\sigma_1 + s\sigma_2 = -\alpha\nu - \gamma\mu, \quad t\sigma_2 = -\alpha\mu - \gamma\nu, \quad (67)$$

which can be put in the form by solving for σ_1 and σ_2 ,

$$\sqrt{1 - \mu^2} = (-\gamma/t + s\alpha/t^2)\mu + (-\alpha/t + s\gamma/t^2)\nu, \quad \sqrt{1 - \nu^2} = -(\alpha/t)\mu - (\gamma/t)\nu.$$

Each of these two equations represents a tilted ellipse in the (μ, ν) -plane centered at the origin. We can let α and γ be sufficiently small so that the rotational angle, from the u -axis, of the first ellipse is small, while the length of its semi-major axis is close to 1 and of its semi-minor axis small. Meanwhile, the second ellipse has a rotational angle close to $\pi/2$ with similar lengths of semi-major and semi-minor axes. It follows that these two ellipses intersect at four points within the unit circle. Hence, $p = 9$ under the generic assumption that $\sigma_1\sigma_2 \neq 0$.

Similar to the preceding case, (67) is perturbed to

$$t\sigma_1 \cos(\psi) + s\sigma_2 \cos(\phi) = -\alpha\nu - \gamma\mu, \quad s\sigma_1 \cos(\zeta) + t\sigma_2 = -\alpha\mu - \gamma\nu.$$

The above argument with ellipses asserts that as long as we keep the three angles small, we do have solutions with s, t, ϕ, ψ, ζ as independent variables, so that once more $\Pi_3 : \tilde{F}_2 \mapsto (\omega_1, \omega_2, \omega_3)$ assumes $(0, 0, 0)$ as a regular value and so $\Pi_3^{-1}((0, 0, 0))$ is of dimension 6.

In summary, generic moduli dimension of type $[3, 4, 12 - n]$ is $9 - n$ for $0 \leq n \leq 4$.

The situation is considerably simplified when $\sigma_1 = 0$ and $\sigma_2 \neq 0$. In (60), we can introduce a rotation on the plane spanned by u_{-3} (the second column) and u_1 (the fifth column) with the new basis vectors u_{-3}^*, u_1^* , where

$$u_{-3} = \cos(\psi) u_{-3}^* + \sin(\psi) u_1^*, \quad u_1 = -\sin(\psi) u_{-3}^* + \cos(\psi) u_1^*,$$

relative to which the second column of c_2 is now zero; note that since the corresponding columns of c_1 is zero because $\sigma_1 = 0$, the process does not change anything else. Similarly, we can cancel the third column by the eighth. After the cancellation, the nontrivial term in the fifth column becomes t and the nontrivial term for the eighth column becomes s , with the second and the third columns zero. We thus obtain, up to range equivalence,

$$(\epsilon_2 \ c_2) = \begin{pmatrix} s \sin(\phi) & 0 & 0 & 0 & 0 & s \cos(\phi) & 0 & 0 \\ 0 & 0 & 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & s \\ 0 & 0 & 0 & t \sin(\xi) & 0 & 0 & t \cos(\xi) & 0 \end{pmatrix}. \quad (68)$$

The Hurwitz condition (62) now gives the constraint

$$s \sigma_2 \cos(\phi) = -\alpha\nu - \gamma\mu, \quad t \sigma_2 \cos(\xi) = -\alpha\mu - \gamma\nu, \quad \mu^2 = 1, \quad \sigma_2 = \sqrt{1 - \nu^2}. \quad (69)$$

Assume $st \neq 0$, By the angle being generic we mean the angle is neither 0 nor π . Then $p = 10$ when ϕ, ξ are generic, since the 4×6 c_2 is of rank 4, and the other 6 dimensions come from w_3 and c_1 ; the moduli dimension is 3. In fact, up to range equivalence, it is a degenerate case of the 7-dimensional moduli discussed below (66). On the other hand, if $\phi = 0$ or π then $\xi = 0$ or π , because, with $\mu^2 = 1$, we can verify via (69) that $s^2\sigma_2^2 = (\alpha\nu + \gamma\mu)^2$ if and only if $t^2\sigma_2^2 = (\alpha\mu + \gamma\nu)^2$; it follows that $p = 8$, which can be brought to the degenerate $[3, 4, 8]$ case in Section 4.4 by range equivalence; the moduli dimension is 2.

Assume $s \neq 0$ and $t = 0$. We have $\gamma^2 = 1 (= \mu^2)$, so that $\alpha^2 = \nu^2$ by the second identity in (69), and so the first identity results in $\cos(\phi) = \pm 1$; thus $\sin(\phi) = 0$. $p = 7$ and the moduli dimension is 1. It can be brought to the degenerate $[3, 4, 7]$ type in Section 4.5 by range equivalence. A similar conclusion holds for the case when $s = 0$ and $t \neq 0$.

Assume $s = t = 0$. Then (61) and (69) implies $\alpha^2 = \gamma^2 = \mu^2 = \nu^2 = 1$, so that $\sigma_2 = 0$, a contradiction.

A parallel argument takes care of the case when $\sigma_1 \neq 0$ and $\sigma_2 = 0$.

Lastly, when $\sigma_1 = \sigma_2 = 0$, (69) gives

$$\alpha\mu + \gamma\nu = 0, \quad \alpha\nu + \gamma\mu = 0, \quad \mu^2 = \nu^2 = 1,$$

so that we may assume

$$\alpha = \gamma, \quad \mu = -\nu = 1;$$

in particular $s = t$. As above, we can cancel the first column in (68) by the sixth one, and the fourth by the seventh, so that, up to range equivalence,

$$(\epsilon_2 \quad c_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & s & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & s \end{pmatrix}. \quad (70)$$

We have $p = 8$ if $s \neq 0$, which can be brought to the degenerate $[3, 4, 8]$ case in Section 4.4 by range equivalence. We come down to the quaternion multiplication when $s = 0$.

The above analysis leads us to the following.

Corollary 4. *There are no orthogonal multiplications of type $[3, 4, p]$ when $p = 5$ or 6.*

Remark 4. *In fact, the corollary also follows from the general results discussed in [22, p. 409]. There is no orthogonal multiplications of type $[3, 4, 5]$, since it can be restricted to type $[3, 4, 4]$. Moreover, the type $[3, 4, 6]$ does not exist either, since it can be extended to type $[3, 6, 6]$, and the Hurwitz-Rodon function $\rho(6) = 2 < 3$, so that the type $[3, 4, 6]$ cannot be attained.*

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