

## TRUE-FALSE QUIZ

1. False;  $\operatorname{div} \mathbf{F}$  is a scalar field.
2. True. (See Definition 16.5.1.)
3. True, by Theorem 16.5.3 and the fact that  $\operatorname{div} \mathbf{0} = 0$ .
4. True, by Theorem 16.3.2.
5. False. See Exercise 16.3.35. (But the assertion is true if  $D$  is simply-connected; see Theorem 16.3.6.)
6. False. See the discussion accompanying Figure 8 on page 1092 [ET 1068].
7. False. For example,  $\operatorname{div}(y\mathbf{i}) = 0 = \operatorname{div}(x\mathbf{j})$  but  $y\mathbf{i} \neq x\mathbf{j}$ .
8. True. Line integrals of conservative vector fields are independent of path, and by Theorem 16.3.3,  $\operatorname{work} = \int_C \mathbf{F} \cdot d\mathbf{r} = 0$  for any closed path  $C$ .
9. True. See Exercise 16.5.24.
10. False.  $\mathbf{F} \cdot \mathbf{G}$  is a scalar field, so  $\operatorname{curl}(\mathbf{F} \cdot \mathbf{G})$  has no meaning.
11. True. Apply the Divergence Theorem and use the fact that  $\operatorname{div} \mathbf{F} = 0$ .
12. False by Theorem 16.5.11, because if it were true, then  $\operatorname{div} \operatorname{curl} \mathbf{F} = 3 \neq 0$ .

## EXERCISES

1. (a) Vectors starting on  $C$  point in roughly the direction opposite to  $C$ , so the tangential component  $\mathbf{F} \cdot \mathbf{T}$  is negative.  
Thus  $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} ds$  is negative.  
(b) The vectors that end near  $P$  are shorter than the vectors that start near  $P$ , so the net flow is outward near  $P$  and  $\operatorname{div} \mathbf{F}(P)$  is positive.
2. We can parametrize  $C$  by  $x = x, y = x^2, 0 \leq x \leq 1$  so  

$$\int_C x \, ds = \int_0^1 x \sqrt{1 + (2x)^2} \, dx = \left[ \frac{1}{12}(1 + 4x^2)^{3/2} \right]_0^1 = \frac{1}{12}(5\sqrt{5} - 1).$$
3. 
$$\begin{aligned} \int_C yz \cos x \, ds &= \int_0^\pi (3 \cos t)(3 \sin t) \cos t \sqrt{(1)^2 + (-3 \sin t)^2 + (3 \cos t)^2} \, dt = \int_0^\pi (9 \cos^2 t \sin t) \sqrt{10} \, dt \\ &= 9\sqrt{10} \left( -\frac{1}{3} \cos^3 t \right)_0^\pi = -3\sqrt{10}(-2) = 6\sqrt{10} \end{aligned}$$
4.  $x = 3 \cos t \Rightarrow dx = -3 \sin t \, dt, y = 2 \sin t \Rightarrow dy = 2 \cos t \, dt, 0 \leq t \leq 2\pi$ , so  

$$\begin{aligned} \int_C y \, dx + (x + y^2) \, dy &= \int_0^{2\pi} [(2 \sin t)(-3 \sin t) + (3 \cos t + 4 \sin^2 t)(2 \cos t)] \, dt \\ &= \int_0^{2\pi} (-6 \sin^2 t + 6 \cos^2 t + 8 \sin^2 t \cos t) \, dt = \int_0^{2\pi} [6(\cos^2 t - \sin^2 t) + 8 \sin^2 t \cos t] \, dt \\ &= \int_0^{2\pi} (6 \cos 2t + 8 \sin^2 t \cos t) \, dt = 3 \sin 2t + \frac{8}{3} \sin^3 t \Big|_0^{2\pi} = 0 \end{aligned}$$
  
Or: Notice that  $\frac{\partial}{\partial y}(y) = 1 = \frac{\partial}{\partial x}(x + y^2)$ , so  $\mathbf{F}(x, y) = \langle y, x + y^2 \rangle$  is a conservative vector field. Since  $C$  is a closed curve,  $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C y \, dx + (x + y^2) \, dy = 0$ .

5.  $\int_C y^3 dx + x^2 dy = \int_{-1}^1 [y^3(-2y) + (1-y^2)^2] dy = \int_{-1}^1 (-y^4 - 2y^2 + 1) dy$   
 $= \left[ -\frac{1}{5}y^5 - \frac{2}{3}y^3 + y \right]_{-1}^1 = -\frac{1}{5} - \frac{2}{3} + 1 - \frac{1}{5} - \frac{2}{3} + 1 = \frac{4}{15}$

6.  $\int_C \sqrt{xy} dx + e^y dy + xz dz = \int_0^1 \left( \sqrt{t^4 \cdot t^2} \cdot 4t^3 + e^{t^2} \cdot 2t + t^4 \cdot t^3 \cdot 3t^2 \right) dt = \int_0^1 (4t^6 + 2te^{t^2} + 3t^9) dt$   
 $= \left[ \frac{4}{7}t^7 + e^{t^2} + \frac{3}{10}t^{10} \right]_0^1 = e - \frac{9}{70}$

7. C:  $x = 1 + 2t \Rightarrow dx = 2dt, y = 4t \Rightarrow dy = 4dt, z = -1 + 3t \Rightarrow dz = 3dt, 0 \leq t \leq 1.$

$$\begin{aligned} \int_C xy dx + y^2 dy + yz dz &= \int_0^1 [(1+2t)(4t)(2) + (4t)^2(4) + (4t)(-1+3t)(3)] dt \\ &= \int_0^1 (116t^2 - 4t) dt = \left[ \frac{116}{3}t^3 - 2t^2 \right]_0^1 = \frac{116}{3} - 2 = \frac{110}{3} \end{aligned}$$

8.  $\mathbf{F}(\mathbf{r}(t)) = (\sin t)(1+t)\mathbf{i} + (\sin^2 t)\mathbf{j}, \mathbf{r}'(t) = \cos t\mathbf{i} + \mathbf{j}$  and

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^\pi ((1+t)\sin t \cos t + \sin^2 t) dt = \int_0^\pi \left( \frac{1}{2}(1+t)\sin 2t + \sin^2 t \right) dt \\ &= \left[ \frac{1}{2} \left( (1+t)(-\frac{1}{2}\cos 2t) + \frac{1}{4}\sin 2t \right) + \frac{1}{2}t - \frac{1}{4}\sin 2t \right]_0^\pi = \frac{\pi}{4} \end{aligned}$$

9.  $\mathbf{F}(\mathbf{r}(t)) = e^{-t}\mathbf{i} + t^2(-t)\mathbf{j} + (t^2 + t^3)\mathbf{k}, \mathbf{r}'(t) = 2t\mathbf{i} + 3t^2\mathbf{j} - \mathbf{k}$  and

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (2te^{-t} - 3t^5 - (t^2 + t^3)) dt = \left[ -2te^{-t} - 2e^{-t} - \frac{1}{2}t^6 - \frac{1}{3}t^3 - \frac{1}{4}t^4 \right]_0^1 = \frac{11}{12} - \frac{4}{e}.$$

10. (a) C:  $x = 3 - 3t, y = \frac{\pi}{2}t, z = 3t, 0 \leq t \leq 1$ . Then

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 [3t\mathbf{i} + (3-3t)\mathbf{j} + \frac{\pi}{2}t\mathbf{k}] \cdot [-3\mathbf{i} + \frac{\pi}{2}\mathbf{j} + 3\mathbf{k}] dt = \int_0^1 [-9t + \frac{3\pi}{2}] dt = \frac{1}{2}(3\pi - 9).$$

$$\begin{aligned} \text{(b)} W &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} (3\sin t\mathbf{i} + 3\cos t\mathbf{j} + t\mathbf{k}) \cdot (-3\sin t\mathbf{i} + \mathbf{j} + 3\cos t\mathbf{k}) dt \\ &= \int_0^{\pi/2} (-9\sin^2 t + 3\cos t + 3t\cos t) dt = \left[ -\frac{9}{2}(t - \sin t \cos t) + 3\sin t + 3(t \sin t + \cos t) \right]_0^{\pi/2} \\ &= -\frac{9\pi}{4} + 3 + \frac{3\pi}{2} - 3 = -\frac{3\pi}{4} \end{aligned}$$

11.  $\frac{\partial}{\partial y}[(1+xy)e^{xy}] = 2xe^{xy} + x^2ye^{xy} = \frac{\partial}{\partial x}[e^y + x^2e^{xy}]$  and the domain of  $\mathbf{F}$  is  $\mathbb{R}^2$ , so  $\mathbf{F}$  is conservative. Thus there

exists a function  $f$  such that  $\mathbf{F} = \nabla f$ . Then  $f_y(x, y) = e^y + x^2e^{xy}$  implies  $f(x, y) = e^y + xe^{xy} + g(x)$  and then  
 $f_x(x, y) = yxe^{xy} + e^{xy} + g'(x) = (1+xy)e^{xy} + g'(x)$ . But  $f_x(x, y) = (1+xy)e^{xy}$ , so  $g'(x) = 0 \Rightarrow g(x) = K$ .

Thus  $f(x, y) = e^y + xe^{xy} + K$  is a potential function for  $\mathbf{F}$ .

12.  $\mathbf{F}$  is defined on all of  $\mathbb{R}^3$ , its components have continuous partial derivatives, and

$\operatorname{curl} \mathbf{F} = (0-0)\mathbf{i} - (0-0)\mathbf{j} + (\cos y - \cos y)\mathbf{k} = 0$ , so  $\mathbf{F}$  is conservative by Theorem 16.5.4. Thus there exists a function  $f$  such that  $\nabla f = \mathbf{F}$ . Then  $f_x(x, y, z) = \sin y$  implies  $f(x, y, z) = x \sin y + g(y, z)$  and then  
 $f_y(x, y, z) = x \cos y + g_y(y, z)$ . But  $f_y(x, y, z) = x \cos y$ , so  $g_y(y, z) = 0 \Rightarrow g(y, z) = h(z)$ . Then  
 $f_z(x, y, z) = x \sin y + h(z)$  implies  $f_z(x, y, z) = h'(z)$ . But  $f_z(x, y, z) = -\sin z$ , so  $h(z) = \cos z + K$ . Thus a potential  
function for  $\mathbf{F}$  is  $f(x, y, z) = x \sin y + \cos z + K$ .

13. Since  $\frac{\partial}{\partial y}(4x^3y^2 - 2xy^3) = 8x^3y - 6xy^2 = \frac{\partial}{\partial x}(2x^4y - 3x^2y^2 + 4y^3)$  and the domain of  $\mathbf{F}$  is  $\mathbb{R}^2$ ,  $\mathbf{F}$  is conservative.

Furthermore  $f(x, y) = x^4y^2 - x^2y^3 + y^4$  is a potential function for  $\mathbf{F}$ .  $t = 0$  corresponds to the point  $(0, 1)$  and  $t = 1$  corresponds to  $(1, 1)$ , so  $\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 1) - f(0, 1) = 1 - 1 = 0$ .

14. Here  $\operatorname{curl} \mathbf{F} = 0$ , the domain of  $\mathbf{F}$  is  $\mathbb{R}^3$ , and the components of  $\mathbf{F}$  have continuous partial derivatives, so  $\mathbf{F}$  is conservative.

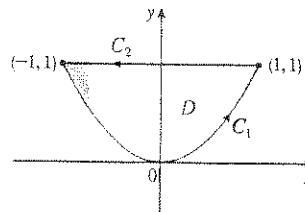
Furthermore  $f(x, y, z) = xe^y + ye^z$  is a potential function for  $\mathbf{F}$ . Then  $\int_C \mathbf{F} \cdot d\mathbf{r} = f(4, 0, 3) - f(0, 2, 0) = 4 - 2 = 2$ .

15.  $C_1: \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, -1 \leq t \leq 1$ ;

$C_2: \mathbf{r}(t) = -t\mathbf{i} + \mathbf{j}, -1 \leq t \leq 1$ .

Then

$$\begin{aligned}\int_C xy^2 dx - x^2y dy &= \int_{-1}^1 (t^5 - 2t^5) dt + \int_{-1}^1 t dt \\ &= [-\frac{1}{6}t^6]_{-1}^1 + [\frac{1}{2}t^2]_{-1}^1 = 0\end{aligned}$$



Using Green's Theorem, we have

$$\begin{aligned}\int_C xy^2 dx - x^2y dy &= \iint_D \left[ \frac{\partial}{\partial x}(-x^2y) - \frac{\partial}{\partial y}(xy^2) \right] dA = \iint_D (-2xy - 2xy) dA = \int_{-1}^1 \int_{x^2}^1 -4xy dy dx \\ &= \int_{-1}^1 [-2xy^2]_{y=x^2}^{y=1} dx = \int_{-1}^1 (2x^5 - 2x) dx = [\frac{1}{3}x^6 - x^2]_{-1}^1 = 0\end{aligned}$$

16.  $\int_C \sqrt{1+x^3} dx + 2xy dy = \iint_D \left[ \frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(\sqrt{1+x^3}) \right] dA = \int_0^1 \int_0^{3x} (2y - 0) dy dx = \int_0^1 9x^2 dx = [3x^3]_0^1 = 3$

17.  $\int_C x^2y dx - xy^2 dy = \iint_{x^2+y^2 \leq 4} \left[ \frac{\partial}{\partial x}(-xy^2) - \frac{\partial}{\partial y}(x^2y) \right] dA = \iint_{x^2+y^2 \leq 4} (-y^2 - x^2) dA = -\int_0^{2\pi} \int_0^2 r^3 dr d\theta = -8\pi$

18.  $\operatorname{curl} \mathbf{F} = (0 - e^{-y} \cos z)\mathbf{i} - (e^{-z} \cos x - 0)\mathbf{j} + (0 - e^{-x} \cos y)\mathbf{k} = -e^{-y} \cos z \mathbf{i} - e^{-z} \cos x \mathbf{j} - e^{-x} \cos y \mathbf{k}$ ,

$$\operatorname{div} \mathbf{F} = -e^{-x} \sin y - e^{-y} \sin z - e^{-z} \sin x$$

19. If we assume there is such a vector field  $\mathbf{G}$ , then  $\operatorname{div}(\operatorname{curl} \mathbf{G}) = 2 + 3z - 2xz$ . But  $\operatorname{div}(\operatorname{curl} \mathbf{F}) = 0$  for all vector fields  $\mathbf{F}$ .

Thus such a  $\mathbf{G}$  cannot exist.

20. Let  $\mathbf{F} = P_1\mathbf{i} + Q_1\mathbf{j} + R_1\mathbf{k}$  and  $\mathbf{G} = P_2\mathbf{i} + Q_2\mathbf{j} + R_2\mathbf{k}$  be vector fields whose first partials exist and are continuous. Then

$$\begin{aligned}\mathbf{F} \operatorname{div} \mathbf{G} - \mathbf{G} \operatorname{div} \mathbf{F} &= \left[ P_1 \left( \frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z} \right) \mathbf{i} + Q_1 \left( \frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z} \right) \mathbf{j} + R_1 \left( \frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z} \right) \mathbf{k} \right] \\ &\quad - \left[ P_2 \left( \frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z} \right) \mathbf{i} + Q_2 \left( \frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z} \right) \mathbf{j} + R_2 \left( \frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z} \right) \mathbf{k} \right]\end{aligned}$$

and

$$\begin{aligned}
 (\mathbf{G} \cdot \nabla) \mathbf{F} - (\mathbf{F} \cdot \nabla) \mathbf{G} = & \left[ \left( P_2 \frac{\partial P_1}{\partial x} + Q_2 \frac{\partial P_1}{\partial y} + R_2 \frac{\partial P_1}{\partial z} \right) \mathbf{i} + \left( P_2 \frac{\partial Q_1}{\partial x} + Q_2 \frac{\partial Q_1}{\partial y} + R_2 \frac{\partial Q_1}{\partial z} \right) \mathbf{j} \right. \\
 & \quad \left. + \left( P_2 \frac{\partial R_1}{\partial x} + Q_2 \frac{\partial R_1}{\partial y} + R_2 \frac{\partial R_1}{\partial z} \right) \mathbf{k} \right] \\
 & - \left[ \left( P_1 \frac{\partial P_2}{\partial x} + Q_1 \frac{\partial P_2}{\partial y} + R_1 \frac{\partial P_2}{\partial z} \right) \mathbf{i} + \left( P_1 \frac{\partial Q_2}{\partial x} + Q_1 \frac{\partial Q_2}{\partial y} + R_1 \frac{\partial Q_2}{\partial z} \right) \mathbf{j} \right. \\
 & \quad \left. + \left( P_1 \frac{\partial R_2}{\partial x} + Q_1 \frac{\partial R_2}{\partial y} + R_1 \frac{\partial R_2}{\partial z} \right) \mathbf{k} \right]
 \end{aligned}$$

Hence

$$\begin{aligned}
 \mathbf{F} \operatorname{div} \mathbf{G} - \mathbf{G} \operatorname{div} \mathbf{F} + (\mathbf{G} \cdot \nabla) \mathbf{F} - (\mathbf{F} \cdot \nabla) \mathbf{G} &= \\
 &= \left[ \left( P_1 \frac{\partial Q_2}{\partial y} + Q_2 \frac{\partial P_1}{\partial x} \right) - \left( P_2 \frac{\partial Q_1}{\partial y} + Q_1 \frac{\partial P_2}{\partial x} \right) \right. \\
 &\quad \left. - \left( P_2 \frac{\partial R_1}{\partial z} + R_1 \frac{\partial P_2}{\partial z} \right) + \left( P_1 \frac{\partial R_2}{\partial z} + R_2 \frac{\partial P_1}{\partial z} \right) \right] \mathbf{i} \\
 &+ \left[ \left( Q_1 \frac{\partial R_2}{\partial z} + R_2 \frac{\partial Q_1}{\partial z} \right) - \left( Q_2 \frac{\partial R_1}{\partial z} + R_1 \frac{\partial Q_2}{\partial z} \right) \right. \\
 &\quad \left. - \left( P_1 \frac{\partial Q_2}{\partial x} + Q_2 \frac{\partial P_1}{\partial x} \right) + \left( P_2 \frac{\partial Q_1}{\partial x} + Q_1 \frac{\partial P_2}{\partial x} \right) \right] \mathbf{j} \\
 &+ \left[ \left( P_2 \frac{\partial R_1}{\partial x} + R_1 \frac{\partial P_2}{\partial x} \right) - \left( P_1 \frac{\partial R_2}{\partial x} + R_2 \frac{\partial P_1}{\partial x} \right) \right. \\
 &\quad \left. - \left( Q_1 \frac{\partial R_2}{\partial y} + R_2 \frac{\partial Q_1}{\partial y} \right) + \left( Q_2 \frac{\partial R_1}{\partial y} + R_1 \frac{\partial Q_2}{\partial y} \right) \right] \mathbf{k} \\
 &= \left[ \frac{\partial}{\partial y} (P_1 Q_2 - P_2 Q_1) - \frac{\partial}{\partial z} (P_2 R_1 - P_1 R_2) \right] \mathbf{i} \\
 &\quad + \left[ \frac{\partial}{\partial z} (Q_1 R_2 - Q_2 R_1) - \frac{\partial}{\partial x} (P_1 Q_2 - P_2 Q_1) \right] \mathbf{j} \\
 &\quad + \left[ \frac{\partial}{\partial x} (P_2 R_1 - P_1 R_2) - \frac{\partial}{\partial y} (Q_1 R_2 - Q_2 R_1) \right] \mathbf{k} \\
 &= \operatorname{curl}(\mathbf{F} \times \mathbf{G})
 \end{aligned}$$

21. For any piecewise-smooth simple closed plane curve  $C$  bounding a region  $D$ , we can apply Green's Theorem to

$$\mathbf{F}(x, y) = f(x) \mathbf{i} + g(y) \mathbf{j} \text{ to get } \int_C f(x) dx + g(y) dy = \iint_D \left[ \frac{\partial}{\partial x} g(y) - \frac{\partial}{\partial y} f(x) \right] dA = \iint_D 0 dA = 0.$$

$$\begin{aligned}
& \frac{\partial x^2}{\partial x^2} + \frac{\partial y^2}{\partial y^2} + \frac{\partial z^2}{\partial z^2} \\
&= \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} g + f \frac{\partial g}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} g + f \frac{\partial g}{\partial y} \right) + \frac{\partial}{\partial z} \left( \frac{\partial f}{\partial z} g + f \frac{\partial g}{\partial z} \right) \quad [\text{Product Rule}] \\
&= \frac{\partial^2 f}{\partial x^2} g + 2 \frac{\partial f}{\partial x} \frac{\partial g}{\partial x} + f \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} g + 2 \frac{\partial f}{\partial y} \frac{\partial g}{\partial y} \\
&\quad + f \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} g + 2 \frac{\partial f}{\partial z} \frac{\partial g}{\partial z} + f \frac{\partial^2 g}{\partial z^2} \quad [\text{Product Rule}] \\
&= f \left( \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2} \right) + g \left( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right) + 2 \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right\rangle \\
&= f \nabla^2 g + g \nabla^2 f + 2 \nabla f \cdot \nabla g
\end{aligned}$$

*Another method:* Using the rules in Exercises 14.6.37(b) and 16.5.25, we have

$$\begin{aligned}
\nabla^2(fg) &= \nabla \cdot \nabla(fg) = \nabla \cdot (g \nabla f + f \nabla g) = \nabla g \cdot \nabla f + g \nabla \cdot \nabla f + \nabla f \cdot \nabla g + f \nabla \cdot \nabla g \\
&= g \nabla^2 f + f \nabla^2 g + 2 \nabla f \cdot \nabla g
\end{aligned}$$

23.  $\nabla^2 f = 0$  means that  $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ . Now if  $\mathbf{F} = f_y \mathbf{i} - f_x \mathbf{j}$  and  $C$  is any closed path in  $D$ , then applying Green's Theorem, we get

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C f_y dx - f_x dy = \iint_D \left[ \frac{\partial}{\partial x} (-f_x) - \frac{\partial}{\partial y} (f_y) \right] dA \\
&= - \iint_D (f_{xx} + f_{yy}) dA = - \iint_D 0 dA = 0
\end{aligned}$$

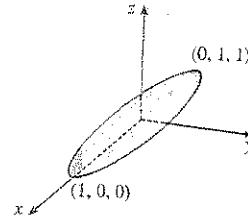
Therefore the line integral is independent of path, by Theorem 16.3.3.

24. (a)  $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$ , so  $C$  lies on the circular cylinder  $x^2 + y^2 = 1$ .

But also  $y = z$ , so  $C$  lies on the plane  $y = z$ . Thus  $C$  is the intersection of the plane  $y = z$  and the cylinder  $x^2 + y^2 = 1$ .

- (b) Apply Stokes' Theorem,  $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ :

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xe^{2y} & 2x^2e^{2y} + 2y \cot z & -y^2 \csc^2 z \end{vmatrix} = \langle -2y \csc^2 z - (-2y \csc^2 z), 0, 4xe^{2y} - 4xe^{2y} \rangle = 0$$



Therefore  $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S 0 \cdot d\mathbf{S} = 0$ .

25.  $z = f(x, y) = x^2 + 2y$  with  $0 \leq x \leq 1$ ,  $0 \leq y \leq 2x$ . Thus

$$A(S) = \iint_D \sqrt{1 + 4x^2 + 4} dA = \int_0^1 \int_0^{2x} \sqrt{5 + 4x^2} dy dx = \int_0^1 2x \sqrt{5 + 4x^2} dx = \left. \frac{1}{6}(5 + 4x^2)^{3/2} \right|_0^1 = \frac{1}{6}(27 - 5\sqrt{5}).$$

26. (a)  $\mathbf{r}_u = -v\mathbf{j} + 2u\mathbf{k}$ ,  $\mathbf{r}_v = 2v\mathbf{i} - u\mathbf{j}$  and

$\mathbf{r}_u \times \mathbf{r}_v = 2u^2\mathbf{i} + 4uv\mathbf{j} + 2v^2\mathbf{k}$ . Since the point  $(4, -2, 1)$  corresponds to  $u = 1, v = 2$  (or  $u = -1, v = -2$  but  $\mathbf{r}_u \times \mathbf{r}_v$  is the same for both), a normal vector to the surface at  $(4, -2, 1)$  is  $2\mathbf{i} + 8\mathbf{j} + 8\mathbf{k}$  and an equation of the tangent plane is  $2x + 8y + 8z = 0$  or  $x + 4y + 4z = 0$ .

(c) By Definition 16.6.6, the area of  $S$  is given by

$$A(S) = \int_0^3 \int_{-3}^3 \sqrt{(2u^2)^2 + (4uv)^2 + (2v^2)^2} dv du = 2 \int_0^3 \int_{-3}^3 \sqrt{u^4 + 4u^2v^2 + v^4} dv du.$$

(d) By Equation 16.7.9, the surface integral is

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^3 \int_{-3}^3 \left\langle \frac{(u^2)^2}{1+(v^2)^2}, \frac{(v^2)^2}{1+(-uv)^2}, \frac{(-uv)^2}{1+(u^2)^2} \right\rangle \cdot \langle 2u^2, 4uv, 2v^2 \rangle dv du \\ &= \int_0^3 \int_{-3}^3 \left( \frac{2u^6}{1+v^4} + \frac{4uv^5}{1+u^2v^2} + \frac{2u^2v^4}{1+u^4} \right) dv du \approx 1524.0190 \end{aligned}$$

27.  $z = f(x, y) = x^2 + y^2$  with  $0 \leq x^2 + y^2 \leq 4$  so  $\mathbf{r}_x \times \mathbf{r}_y = -2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}$  (using upward orientation). Then

$$\begin{aligned} \iint_S z \, dS &= \iint_{x^2+y^2 \leq 4} (x^2 + y^2) \sqrt{4x^2 + 4y^2 + 1} \, dA \\ &= \int_0^{2\pi} \int_0^2 r^2 \sqrt{1+4r^2} \, dr \, d\theta = \frac{1}{60}\pi(391\sqrt{17} + 1) \end{aligned}$$

(Substitute  $u = 1 + 4r^2$  and use tables.)

28.  $z = f(x, y) = 4 + x + y$  with  $0 \leq x^2 + y^2 \leq 4$  so  $\mathbf{r}_x \times \mathbf{r}_y = -\mathbf{i} - \mathbf{j} + \mathbf{k}$ . Then

$$\begin{aligned} \iint_S (x^2z + y^2z) \, dS &= \iint_{x^2+y^2 \leq 4} (x^2 + y^2)(4 + x + y) \sqrt{3} \, dA \\ &= \int_0^2 \int_0^{2\pi} \sqrt{3}r^3(4 + r\cos\theta + r\sin\theta) \, d\theta \, dr = \int_0^2 8\pi\sqrt{3}r^3 \, dr = 32\pi\sqrt{3} \end{aligned}$$

29. Since the sphere bounds a simple solid region, the Divergence Theorem applies and

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E \operatorname{div} \mathbf{F} \, dV = \iiint_E (z - 2) \, dV = \iiint_E z \, dV - 2 \iiint_E \, dV \\ &= 0 \left[ \begin{array}{l} \text{odd function in } z \\ \text{and } E \text{ is symmetric} \end{array} \right] - 2 \cdot V(E) = -2 \cdot \frac{4}{3}\pi(2)^3 = -\frac{64}{3}\pi \end{aligned}$$

Alternate solution:  $\mathbf{F}(\mathbf{r}(\phi, \theta)) = 4\sin\phi\cos\theta\cos\phi\mathbf{i} - 4\sin\phi\sin\theta\mathbf{j} + 6\sin\phi\cos\theta\mathbf{k}$ ,

$\mathbf{r}_\phi \times \mathbf{r}_\theta = 4\sin^2\phi\cos\theta\mathbf{i} + 4\sin^2\phi\sin\theta\mathbf{j} + 4\sin\phi\cos\phi\mathbf{k}$ , and

$\mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) = 16\sin^3\phi\cos^2\theta\cos\phi - 16\sin^3\phi\sin^2\theta + 24\sin^2\phi\cos\phi\cos\theta$ . Then

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi} \int_0^\pi (16\sin^3\phi\cos\phi\cos^2\theta - 16\sin^3\phi\sin^2\theta + 24\sin^2\phi\cos\phi\cos\theta) \, d\phi \, d\theta \\ &= \int_0^{2\pi} \frac{4}{3}(-16\sin^2\theta) \, d\theta = -\frac{64}{3}\pi \end{aligned}$$

